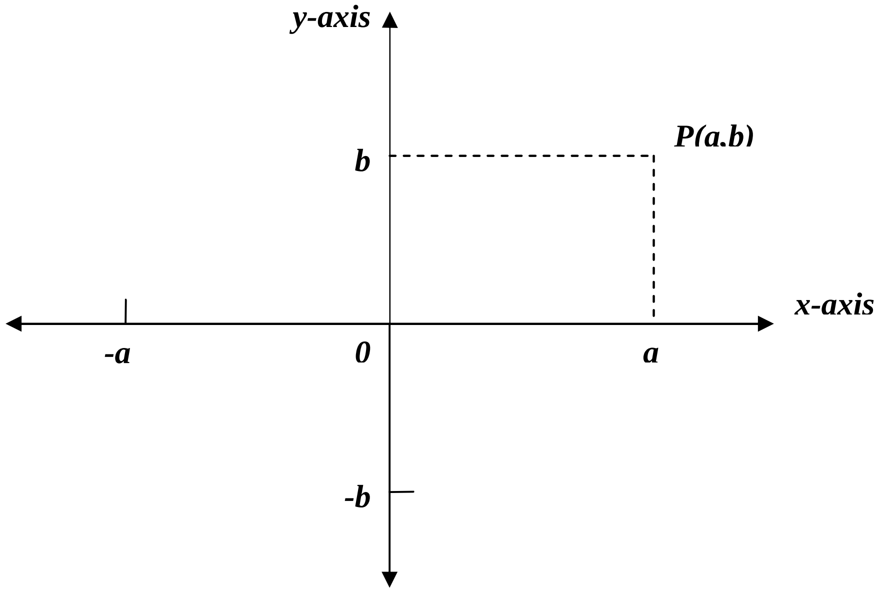
Chapter – one
The Rate of Change of a Function

1-1- Coordinates for the plane :

***Cartesian Coordinate*- Two number lines , one of them horizontal (called *x-axis*) and the other vertical (called *y-axis*). The point where the lines cross is the *origin* . Each line is assumed to represent the real number .**

On the *x-axis* , the positive number *a* lies *a* units to the right of the *origin* , and the negative number *–a* lies *a* units to the left of the *origin* . On the *y-axis* , the positive number *b* lies *b* units above the *origin* while the negative where *–b* lies *b* units below the *origin* .

With the axes in place , we assign a pair *(a,b)* of real number to each point *P* in the plane . The number *a* is the number at the foot of the perpendicular from *P* to the *x-axis* (called *x-coordinate of P*). The number *b* is the number at the foot of the perpendicular from *P* to the *y-axis* (called *y-coordinate of P*).



1-2- The Slope of a line :

***Increments* – When a particle moves from one position in the plane to another , the net changes in the particle's coordinates are calculated by subtracting the coordinates of the starting point (x_1, y_1) from the coordinates of the stopping point (x_2, y_2) ,**

i.e. $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$.

Slopes of nonvertical lines :

Let *L* be a nonvertical line in the plane ,

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two points on *L*.

Then the slope *m* is :

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{where} \quad \Delta x \neq 0$$

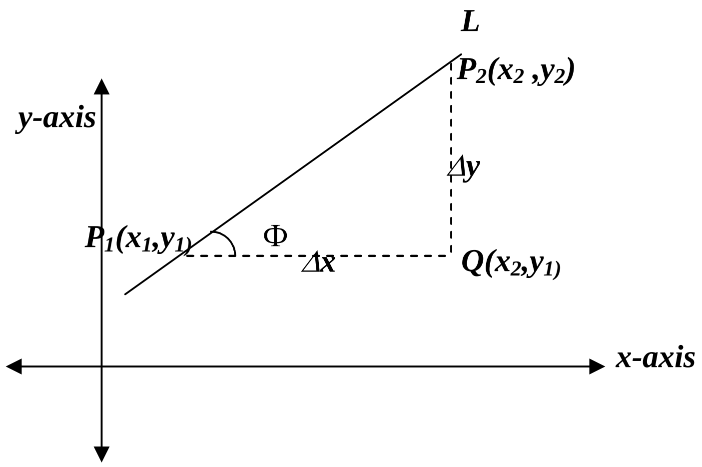
- A line that goes uphill as x increases has a positive slope . A line that goes downhill as x increases has a negative slope .
- A horizontal line has slope zero because $\Delta y = 0$.
- The slope of a vertical line is undefined because $\Delta x = 0$.
- Parallel lines have same slope .
- If neither of two perpendicular lines L_1 and L_2 is vertical , their slopes m_1 and m_2 are related by the equation : $m_1 . m_2 = -1$.

Angles of Inclination: The angle of inclination of a line that crosses the *x-axis* is the smallest angle we get when we measure counter clock from the *x-axis* around the point of intersection .

The slope of a line is the tangent of the line angle of inclination .

$$m = \tan \Phi \qquad \text{where } \Phi \text{ is the angle of inclination .}$$

- The angle of inclination of a horizontal line is taken to be 0^o .
- Parallel lines have equal angle of inclination .



EX-1- Find the slope of the line determined by two points *A(2,1)* and *B(-1,3)* and find the common slope of the line perpendicular to *AB*.

Sol.- Slope of *AB* is: $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-1-2} = -\frac{2}{3}$

Slope of line perpendicular to *AB* is : $-\frac{1}{m_{AB}} = \frac{3}{2}$

EX-2- Use slopes to determine in each case whether the points are collinear (lie on a common straight line) :

a) *A(1,0) , B(0,1) , C(2,1)* .

b) *A(-3,-2) , B(-2,0) , C(-1,2) , D(1,6)* .

Sol. –

a) $m_{AB} = \frac{1-0}{0-1} = -1$ and $m_{BC} = \frac{1-1}{2-0} = 0 \neq m_{AB}$

The points *A* , *B* and *C* are not lie on a common straight line .

b) $m_{AB} = \frac{0-(-2)}{-2-(-3)} = 2$, $m_{BC} = \frac{2-0}{-1-(-2)} = 2$, $m_{CD} = \frac{6-2}{1-(-1)} = 2$

Since $m_{AB} = m_{BC} = m_{CD}$

Hence the points *A* , *B* , *C* , and *D* are collinear .

1-3- Equations for lines : An equation for a line is an equation that is satisfied by the coordinates of the points that lies on the line and is not satisfied by the coordinates of the points that lie elsewhere .

Vertical lines : Every vertical line *L* has to cross the x-axis at some point *(a,0)*. The other points on *L* lie directly above or below *(a,0)* . This mean that : $x = a \quad \forall (x, y)$

Nonvertical lines **:** That point – slope equation of the line through the point (x_1, y_1) with slope *m* is :

$$y - y_1 = m(x - x_1)$$

Horizontal lines **:** The standard equation for the horizontal line through the point (a, b) is : $y = b$.

The distance from a point to a line **:** To calculate the distance *d* between the point $P(x_1, y_1)$ and $Q(x_2, y_2)$ is :

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

We use this formula when the coordinate axes are scaled in a common unit .

To find the distance from the point $P(x_1, y_1)$ to the line *L* , we follow :

1. Find an equation for the line *L'* through *P* perpendicular to *L* :

   $y - y_1 = m'(x - x_1)$ where $m' = -1/m$

2. Find the point $Q(x_2, y_2)$ by solving the equation for *L* and *L'* simultaneously .

3. Calculate the distance between *P* and *Q* .

The general linear equation :

$$Ax + By = C$$ where *A* and *B* not both zero.

EX-3 – Write an equation for the line that passes through point :

a) $P(-1, 3)$ with slope $m = -2$.

b) $P_1(-2, 0)$ and $P_2(2, -2)$.

Sol. - a) $y - y_1 = m(x - x_1) \rightarrow y - 3 = -2(x - (-1)) \rightarrow y + 2x = 1$

b)

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-0}{2-(-2)} = -\frac{1}{2}$$

$$y - y_1 = m(x - x_1) \Rightarrow \quad y - 0 = -\frac{1}{2}(x - (-2)) \Rightarrow \quad 2y + x + 2 = 0$$

EX-4 - Find the slope of the line : $3x + 4y = 12$.

Sol. - $y = -\frac{3}{4}x + 3 \Rightarrow \quad the \quad slope \quad is \quad m = -\frac{3}{4}$

EX-5- Find :

a) an equation for the line through $P(2,1)$ parallel to $L: \; y = x + 2$.

b) an equation for the line through P perpendicular to L .

c) the distance from P to L .

Sol.-

a)

$$since \quad L_2 \,//\, L_1 \Rightarrow \quad m_{L2} = m_{L1} = 1 \Rightarrow \quad y - 1 = 1(x - 2) \quad \Rightarrow \quad y = x - 1$$

b) Since L_1 and L_3 are perpendicular lines then :

$$m_{L3} = -1 \Rightarrow \quad y - 1 = -(x - 2) \Rightarrow \quad y + x = 3$$

c)

$$\begin{matrix} y = x + 2 \\ y + x = 3 \end{matrix} \quad \Rightarrow \quad x = \frac{1}{2} \quad and \quad y = \frac{5}{2} \quad \Rightarrow \quad P(2,1) \quad and \quad Q\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$\Rightarrow d = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} = \sqrt{4.5}$$

EX-6 – Find the angle of inclination of the line : $\sqrt{3}x + y = -3$

Sol.-

$$y = -\sqrt{3}x - 3 \quad \Rightarrow \quad m = -\sqrt{3}$$

$$m = \tan\Phi = -\sqrt{3} \quad \Rightarrow \quad \Phi = 120^o$$

EX-7- Find the line through the point $P(1, 4)$ with the angle of inclination $\Phi = 60^o$.

Sol.-

$$m = \tan\Phi = \tan 60 = \sqrt{3}$$

$$y - 4 = \sqrt{3}(x - 1) \Rightarrow \quad y = \sqrt{3}x + 4 - \sqrt{3}$$

EX-8- The pressure P experienced by a diver under water is related to the diver's depth d by an equation of the form $P = k\,d + 1$ where k a constant . When $d = 0$ meters , the pressure is 1 atmosphere . The pressure at 100 meters is about 10.94 atmosphere . Find the pressure at 50 meters.

Sol.- At $P = 10.94$ and $d = 100 \;\rightarrow\; 10.94 = k(100)+1 \rightarrow k = 0.0994$

$P = 0.0994\,d + 1$, at $d = 50 \;\rightarrow\; P = 0.0994 * 50 + 1 = 5.97$ atmo.

1-4- <u>Functions</u> : ***Function*** **is any rule that assigns to each element in one set some element from another set :**

$$y = f(x)$$

The inputs make up the ***domain of the function*** **, and the outputs make up** ***the function's range.***

The variable x **is called** ***independent variable of the function*** **, and the variable** y **whose value depends on** x **is called** ***the dependent variable of the function .***

We must keep two restrictions in mind when we define functions :

1. **We never divide by zero .**
2. **We will deal with real – valued functions only.**

<u>Intervals</u> :

- **The** ***open interval*** **is the set of all real numbers that be strictly between two fixed numbers** a **and** b **:**

$$(a,b) \equiv a < x < b$$

- **The** ***closed interval*** **is the set of all real numbers that contain both endpoints :**

$$[a,b] \equiv a \le x \le b$$

- ***Half open interval*** **is the set of all real numbers that contain one endpoint but not both :**

$$[a,b) \equiv a \le x < b$$

$$(a,b] \equiv a < x \le b$$

<u>Composition of functions</u> : **suppose that the outputs of a function** f **can be used as inputs of a function** g **. We can then hook** f **and** g **together to form a new function whose inputs are the inputs of** f **and whose outputs are the numbers :**

$$(g_o f)(x) = g(f(x))$$

***<u>EX-9</u>*- Find the domain and range of each function :**

$a)\quad y = \sqrt{x+4} \quad , \quad b)\quad y = \dfrac{1}{x-2}$

$c)\quad y = \sqrt{9 - x^2} \quad , \quad d)\quad y = \sqrt{2 - \sqrt{x}}$

<u>Sol.</u> - $a)\quad x + 4 \ge 0 \Rightarrow \quad x \ge -4 \Rightarrow \quad D_x : \forall x \ge -4 \quad , \quad R_y : \forall y \ge 0$

$b)\quad x - 2 \ne 0 \Rightarrow \quad x \ne 2 \Rightarrow \quad D_x : \forall x \ne 2$

$$y = \frac{1}{x-2} \Rightarrow \quad x = \frac{1}{y} + 2 \Rightarrow \quad R_y : \forall y \ne 0$$

$c)\quad 9 - x^2 \ge 0 \Rightarrow \quad -3 \le x \le 3 \Rightarrow \quad D_x : -3 \le x \le 3$

$$y = \sqrt{9 - x^2} \Rightarrow \quad x = \mp\sqrt{9 - y^2}$$

$$since \quad 9 - y^2 \ge 0 \Rightarrow \quad -3 \le y \le 3$$

$$since \quad y \ge 0 \quad \Rightarrow \quad R_y : 0 \le y \le 3$$

$$d) \quad 2-\sqrt{x}\ge 0 \Rightarrow \quad 0\le x\le 4 \Rightarrow \quad D_x : 0\le x\le 4$$

$$\begin{matrix} if & x=0\Rightarrow & y=\sqrt{2} \\ if & x=4\Rightarrow & y=0 \end{matrix} \quad \Rightarrow \quad R_y : 0\le y\le \sqrt{2}$$

EX-10- Let $f(x)=\dfrac{x}{x-1}$ *and* $g(x)=1+\dfrac{1}{x}$.

Find $(g_o f)(x)$ **and** $(f_o g)(x)$.

Sol.-

$$(g_o f)(x)=g(f(x))=g\left(\frac{x}{x-1}\right)=1+\frac{1}{\frac{x}{x-1}}=\frac{2x-1}{x}$$

$$(f_o g)(x)=f(g(x))=f\left(1+\frac{1}{x}\right)=\frac{1+\frac{1}{x}}{1+\frac{1}{x}-1}=x+1$$

EX-11- **Let** $(g_o f)(x)=x$ *and* $f(x)=\dfrac{1}{x}$. **Find** $g(x)$.

Sol.- $(g_o f)(x)=g\left(\dfrac{1}{x}\right)=x\Rightarrow \quad g(x)=\dfrac{1}{x}$

1-5- Limits and continuity :

Limits **:** **The limit of** $F(t)$ **as** t **approaches** C **is the number** L **if :**
Given any radius $\varepsilon>0$ **about** L **there exists a radius** $\delta>0$ **about** C **such that for all** t**,** $0<|t-C|<\delta$ **implies** $|F(t)-L|<\varepsilon$ **and we can write it as :**

$$\lim_{t\to C} F(t)=L$$

The limit of a function $F(t)$ **as** $t\to C$ **never depend on what happens when** $t=C$ **.**

Right hand limit **:** $\lim_{t\to C^+} F(t)=L$

The limit of the function $F(t)$ **as** $t\to C$ **from the right equals** L **if :**
Given any $\varepsilon>0$ **(radius about** L **) there exists a** $\delta>0$ **(radius to the right of** C **) such that for all** t **:**

$$C<t<C+\delta\Rightarrow \quad |F(t)-L|<\varepsilon$$

Left hand limit **:** $\lim_{t\to C^-} F(t)=L$

The limit of the function $F(t)$ **as** $t\to C$ **from the left equal** L **if :**
Given any $\varepsilon>0$ **there exists a** $\delta>0$ **such that for all** t **:**

$$C-\delta<t<C\Rightarrow \quad |F(t)-L|<\varepsilon$$

Note that – A function $F(t)$ has a limit at point C if and only if the right hand and the left hand limits at C exist and equal . In symbols :

$$\lim_{t\to C} F(t) = L \quad \Leftrightarrow \quad \lim_{t\to C^+} F(t) = L \quad and \quad \lim_{t\to C^-} F(t) = L$$

The limit combinations theorems :

$$1)\quad \lim[F_1(t) \mp F_2(t)] = \lim F_1(t) \mp \lim F_2(t)$$

$$2)\quad \lim[F_1(t) * F_2(t)] = \lim F_1(t) * \lim F_2(t)$$

$$3)\quad \lim \frac{F_1(t)}{F_2(t)} = \frac{\lim F_1(t)}{\lim F_2(t)} \qquad where \quad \lim F_2(t) \neq 0$$

$$4)\quad \lim[k * F_1(t)] = k * \lim F_1(t) \qquad \forall k$$

$$5)\quad \lim_{\theta\to 0} \frac{Sin\theta}{\theta} = 1$$

provided that θ *is measured in radius*

The limits (in *1 – 4*) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

Thm. -1 : The sandwich theorem : Suppose that $f(t) \le g(t) \le h(t)$ for all $t \neq C$ in some interval about C and that $f(t)$ and $h(t)$ approaches the same limit L as $t \to C$, then :

$$\lim_{t\to C} g(t) = L$$

Infinity as a limit :

1.The limit of the function $f(x)$ as x approaches infinity is the number L: $\lim_{x\to\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number M such that for all x : $M < x \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

2. The limit of $f(x)$ as x approaches negative infinity is the number L : $\lim_{x\to-\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number N such that for all x : $x < N \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

The following facts are some times abbreviated by saying :

a) As x approaches 0 from the right , $1/x$ tends to ∞ .

b) As x approaches 0 from the left , $1/x$ tends to $-\infty$.

c) As x tends to ∞ , $1/x$ approaches 0 .

d) As x tends to $-\infty$, $1/x$ approaches 0 .

Continuity :

Continuity at an interior point : A function $y = f(x)$ is continuous at an interior point C of its domain if : $\lim_{x\to C} f(x) = f(C)$.

Continuity at an endpoint : A function $y = f(x)$ is continuous at a left endpoint a of its domain if : $\lim_{x\to a^+} f(x) = f(a)$.

A function $y = f(x)$ is continuous at a right endpoint b of its domain if: $\lim_{t\to b^-} f(x) = f(b)$.

Continuous function **:** **A function is continuous if it is continuous at each point of its domain .**

Discontinuity at a point **: If a function f is not continuous at a point C, we say that f is discontinuous at C, and call C a point of discontinuity of f .**

The continuity test **: The function $y = f(x)$ is continuous at $x = C$ if and only if all three of the following statements are true :**

1) **$f(C)$ exist (C is in the domain of f) .**
2) **$\lim\limits_{x\to C} f(x)$ exists (f has a limit as $x\to C$) .**
3) **$\lim\limits_{x\to C} f(x) = f(C)$ (the limit equals the function value) .**

Thm.-2 : The limit combination theorem for continuous function :

If the function f and g are continuous at $x = C$, then all of the following combinations are continuous at $x = C$:

$$1)\ f \mp g \quad 2)\ f.g \quad 3)\ k.g \ \ \forall k \quad 4)\ g_o f\ ,\ f_o g \quad 5)\ f/g$$

$$provided \quad g(C) \neq 0$$

Thm.-3 **: A function is continuous at every point at which it has a derivative . That is , if $y = f(x)$ has a derivative $f'(C)$ at $x = C$, then f is continuous at $x = C$.**

EX-12 **– Find :**

$$1)\ \lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2} \quad , \quad 2)\ \lim_{x\to a}\frac{x^3-a^3}{x^4-a^4}$$

$$3)\ \lim_{x\to 0}\frac{Sin5x}{Sin3x} \quad , \quad 4)\ \lim_{y\to 0}\frac{tan2y}{3y}$$

$$5)\ \lim_{x\to 0}\frac{Sin2x}{2x^2+x} \quad , \quad 6)\ \lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)$$

$$7)\ \lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5} \quad , \quad 8)\ \lim_{y\to\infty}\frac{3y+7}{y^2-2}$$

$$9)\ \lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5} \quad , \quad 10)\ \lim_{x\to -1^-}\frac{1}{x+1}$$

$$11)\ \lim_{x\to 0}Cos\left(1-\frac{Sinx}{x}\right) \quad , \quad 12)\ \lim_{x\to 0}Sin\left(\frac{\pi}{2}Cos(tan\,x)\right)$$

S0l.-

$$1)\ \lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}=\lim_{x\to 0}\frac{5x+8}{3x^2-16}=\frac{0+8}{0-16}=-\frac{1}{2}$$

$$2)\ \lim_{x\to a}\frac{x^3-a^3}{x^4-a^4}=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)(x^2+a^2)}=\frac{a^2+a^2+a^2}{(a+a)(a^2+a^2)}=\frac{3}{4a}$$

$$3)\ \lim_{x\to 0}\frac{5\dfrac{Sin5x}{5x}}{3\dfrac{Sin3x}{3x}}=\frac{5}{3}.\frac{\lim\limits_{5x\to 0}\dfrac{Sin5x}{5x}}{\lim\limits_{3x\to 0}\dfrac{Sin3x}{3x}}=\frac{5}{3}$$

$$4)\ \lim_{y\to 0}\frac{\tan 2y}{3y}=\frac{2}{3}.\lim_{2y\to 0}\frac{Sin2y}{2y}.\lim_{y\to 0}\frac{1}{Cos2y}=\frac{2}{3}$$

$$5)\ \lim_{x\to 0}\frac{Sin2x}{2x^2+x}=2\lim_{2x\to 0}\frac{Sin2x}{2x}.\lim_{x\to 0}\frac{1}{2x+1}=2$$

$$6)\ \lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)=1+Cos0=2$$

$$7)\ \lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5}=\lim_{x\to\infty}\frac{3+\frac{5}{x}-\frac{7}{x^3}}{10-\frac{11}{x^2}+\frac{5}{x^3}}=\frac{3}{10}$$

$$8)\ \lim_{y\to\infty}\frac{3y+7}{y^2-2}=\lim_{y\to\infty}\frac{\frac{3}{y}+\frac{7}{y^2}}{1-\frac{2}{y^2}}=\frac{0}{1}=0$$

$$9)\ \lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5}=\lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{\frac{2}{x}-\frac{7}{x^2}+\frac{5}{x^3}}=\frac{1}{0}=\infty$$

$$10)\ \lim_{x\to -1^-}\frac{1}{x+1}=\frac{1}{-1+1}=-\infty$$

$$11)\ \lim_{x\to 0}Cos\left(1-\frac{Sinx}{x}\right)=Cos\left(1-\lim_{x\to 0}\frac{Sinx}{x}\right)=Cos0=1$$

$$12)\ \lim_{x\to 0}Sin\left(\frac{\pi}{2}Cos(\tan x)\right)=Sin\left(\frac{\pi}{2}Cos(\tan 0)\right)=Sin\left(\frac{\pi}{2}Cos0\right)=Sin\frac{\pi}{2}=1$$

<u>*EX-13*</u>- Test continuity for the following function :

$$f(x)=\begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0\le x<1 \\ 1 & x=1 \\ -2x+4 & 1<x\le 2 \\ 0 & 2<x\le 3 \end{cases}$$

<u>*Sol.*</u>- We test the continuity at midpoints *x = 0 , 1 , 2* and endpoints *x = -1 , 3 .*

$At\quad x=0\Rightarrow\qquad f(0)=2*0=0$

$$\lim_{x\to 0^-}f(x)=\lim_{x\to 0}(x^2-1)=-1$$

$$\lim_{x\to 0^+}f(x)=\lim_{x\to 0}2x=0\neq\lim_{x\to 0^-}f(x)$$

$$Since\quad \lim_{x\to 0}f(x)\ doesn't\ exist$$

$$Hence\quad the\ function\ discontinuous\ at\ x=0$$

$At \quad x = 1 \Rightarrow \quad f(1) = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} 2x = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(-2x+4) = 2 = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} f(x)$$

$$Since \quad \lim_{x\to 1} f(x) \neq f(1)$$

Hence the function is discontinuous at $x = 1$

$At \quad x = 2 \Rightarrow \quad f(2) = -2 * 2 + 4 = 0$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2}(-2x+4) = 0$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2} 0 = 0 = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x)$$

$$Since \quad \lim_{x\to 2} f(x) = f(2) = 0$$

Hence the function is continuous at $x = 2$

$At \quad x = -1 \Rightarrow \quad f(-1) = (-1)^2 - 1 = 0$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1}(x^2 - 1) = 0 = f(-1)$$

Hence the function is continuous at $x = -1$

$At \quad x = 3 \Rightarrow \quad f(3) = 0$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} 0 = 0 = f(3)$$

Hence the function is continuous at $x = 3$

<u>*EX-14*</u>- What value should be assigned to *a* to make the function :

$$f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases}$$ **continuous at *x = 3* ?**

<u>Sol.</u> –

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \Rightarrow \lim_{x\to 3}(x^2 - 1) = \lim_{x\to 3} 2ax \Rightarrow \quad 8 = 6a \Rightarrow \quad a = \frac{4}{3}$$

Problems – 1

1. **The steel in railroad track expands when heated . For the track temperature encountered in normal outdoor use , the length *S* of a piece of track is related to its temperature *t* by a linear equation . An experiment with a piece of track gave the following measurements :**

$$t_1 = 65^o F \quad , \quad S_1 = 35 \ ft$$

$$t_2 = 135^o F \quad , \quad S_2 = 35.16 \ ft$$

Write a linear equation for the relation between *S* and *t* .

(ans.: S=0.0023t+34.85)

2. **Three of the following four points lie on a circle center the origin . Which are they , and what is the radius of the circle ?**
   A(-1.7) , B(5,-5) , C(-7,5) and D(7,-1). *(ans.: A,B,D;√50)*

3. ***A* and *B* are the points *(3,4)* and *(7,1)* respectively . Use Pythagoras theorem to prove that *OA* is perpendicular to *AB* . Calculate the slopes of *OA* and *AB* , and find their product .** *(ans.: 4/3, -3/4;-1)*

4. ***P(-2,-4)* , *Q(-5,-2)* , *R(2,1)* and *S* are the vertices of a parallelogram . Find the coordinates of *M* , the point of intersection of the diagonals and of *S*.**
   (ans.: M(0,-3/2) , S(5,-1))

5. **Calculate the area of the triangle formed by the line $3x-7y+4=0$, and the axes .** *(ans.: 8/21)*

6. **Find the equation of the straight line through *P(7,5)* perpendicular to the straight line *AB* whose equation is $3x + 4y - 16 = 0$. Calculate the length of the perpendicular from *P* and *AB*.** *(ans.: 3y-4x+13=0;5)*

7. ***L(-1,0)* , *M(3,7)* and *N(5,-2)* are the mid-points of the sides *BC* , *CA* and *AB* respectively of the triangle *ABC*. Find the equation of *AB*.** *(ans.:4y=7x-43)*

8. **The straight line $x - y - 6 = 0$ cuts the curve $y^2 = 8x$ at *P* and *Q* . Calculate the length of *PQ* .** *(ans.:16√2)*

9. **A line is drawn through the point *(2,3)* making an angle of 45^o with the positive direction of the x-axis and it meets the line $x = 6$ at *P* . Find the distance of *P* from the origin *O* , and the equation of the line through *P* perpendicular to *OP*.** *(ans.: √85,7y+6x-85=0)*

10. **The vertices of a quadrilateral *ABCD* are *A(4,0)* , *B(14,11)* , *C(0,6)* and *D(-10,-5)* . Prove that the diagonals *AC* and *BD* bisect each other at right angles , and that the length of *BD* is four times that of *AC* .**

11. **The coordinates of the vertices** A, B **and** C **of the triangle** ABC **are** $(-3,7)$ **,** $(2,19)$ **and** $(10,7)$ **respectively :**

a) Prove that the triangle is isosceles.

b) Calculate the length of the perpendicular from B **to** AC **, and use it to find the area of the triangle .** ***(ans.:12,78)***

12. **Find the equations of the lines which pass through the point of intersection of the lines** $x - 3y = 4$ **and** $3x + y = 2$ **and are respectively parallel and perpendicular to the line** $3x + 4y = 0$ **.**

(ans.:4y+3x+1=0;3y-4x+7=0)

13. **Through the point** $A(1,5)$ **is drawn a line parallel to the x-axis to meet at** B **the line** PQ **whose equation is** $3y = 2x - 5$ **. Find the length of** AB **and the sine of the angle between** PQ **and** AB **; hence show that the length of the perpendicular from** A **to** PQ **is** $18/\sqrt{13}$ **. Calculate the area of the triangle formed by** PQ **and the axes .** ***(ans.:9,2/√13,25/12)***

14. **Let** $y = \frac{x^2 + 2}{x^2 - 1}$ **, express** x **in terms of** y **and find the values of** y **for which** x **is real .** **(ans.:** $x = \mp\sqrt{\frac{y+2}{y-1}}; y \le -2 \text{ or } y > 1$**)**

15. **Find the domain and range of each function :**

$$a)\ y = \frac{1}{1+x^2} \quad , \quad b)\ y = \frac{1}{1+\sqrt{x}} \quad , \quad c)\ y = \frac{1}{\sqrt{3-x}}$$

$$(ans.: a)\forall x, 0 < y \le 1;\ b) x \ge 0, y > 0;\ c) x < 3, y > 0)$$

16. **Find the points of intersection of** $x^2 = 4y$ **and** $y = 4x$ **.** ***(ans.:(0,0),(16,64))***

17. **Find the coordinates of the points at which the curves cut the axes :**

$$a)\ y = x^3 - 9x^2 \quad , \quad b)\ y = (x^2 - 1)(x^2 - 9) \quad , \quad c)\ y = (x+1)(x-5)^2$$

(ans.:a)(0,0);(0,0),(9,0);b)(0,9);(1,0),(-1,0),(3,0),(-3,0);c)(0,25);(-1,0),(5,0))

18. **Let** $f(x) = ax + b$ **and** $g(x) = cx + d$ **. What condition must be satisfied by the constants** a **,** b **,** c **and** d **to make** $f(g(x))$ **and** $g(f(x))$ **identical ?**

(ans.:ad+b=bc+d)

19. **A particle moves in the plane from** $(-2,5)$ **to the y-axis in such away that** $\Delta y = 3*\Delta x$ **. Find its new coordinates .** ***(ans.:(0,11),(0,-1))***

20. **If** $f(x) = 1/x$ **and** $g(x)=1/\sqrt{x}$ **, what are the domain of** f **,** g **,** $f+g$ **,** $f-g$ **,** $f.g$ **,** f/g **,** g/f **,** $f_o g$ **and** $g_o f$ **? What is the domain of** $h(x) = g(x+4)$ **?**

$$(ans.: \forall x \ne 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x \ge 0, \forall x \ge 0, \forall x \ge 0; \forall x > -4)$$

Chapter two
Functions

2-1- Exponential and Logarithm functions :

Exponential functions : **If a is a positive number and x is any number , we define the exponential function as :**

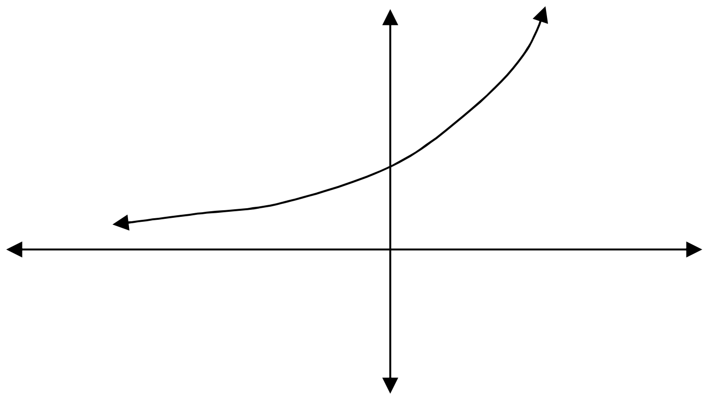
$$y = a^x \quad \text{with domain : } -\infty < x < \infty$$

$$\text{Range : } y > 0$$

The properties of the exponential functions are :

1. If $a > 0 \leftrightarrow a^x > 0$.
2. $a^x \cdot a^y = a^{x+y}$.
3. $a^x / a^y = a^{x-y}$.
4. $(a^x)^y = a^{x.y}$.
5. $(a \cdot b)^x = a^x \cdot b^x$.
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$.
7. $a^{-x} = 1 / a^x$ and $a^x = 1 / a^{-x}$.
8. $a^x = a^y \leftrightarrow x = y$.
9. $a^0 = 1$,
   $a^\infty = \infty$, $a^{-\infty} = 0$, where $a > 1$.
   $a^\infty = 0$, $a^{-\infty} = \infty$, where $a < 1$.

The graph of the exponential function $y = a^x$ is :



Logarithm function : **If a is any positive number other than 1 , then the logarithm of x to the base a denoted by :**

$$y = \log_a x \quad \text{where } x > 0$$

At $a = e = 2.7182828\ldots$, we get the natural logarithm and denoted by :

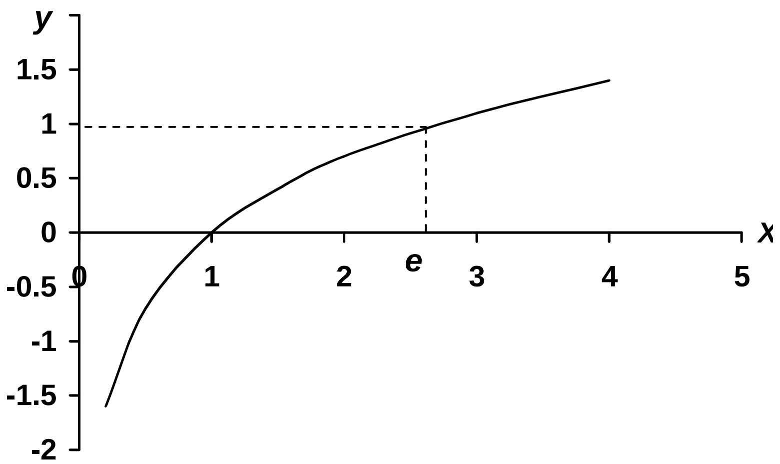
$$y = \ln x$$

Let $x , y > 0$ then the properties of logarithm functions are :

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.

4. $\ln(x.y) = \ln x + \ln y$.
5. $\ln(x/y) = \ln x - \ln y$.
6. $\ln x^n = n.\ln x$.
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x.\ln a}$.
9. $e^{\ln x} = x$.

The graph of the function $y = \ln x$ is :



Application of exponential and logarithm functions **:**

We take Newton's law of cooling :

$$T - T_S = (T_0 - T_S)\, e^{tk}$$

where T is the temperature of the object at time t .

T_S is the surrounding temperature .

T_0 is the initial temperature of the object .

k is a constant .

***EX-1*- The temperature of an ingot of metal is $80\ ^oC$ and the room temperature is $20\ ^oC$. After twenty minutes, it was $70\ ^oC$.**

a) **What is the temperature will the metal be after 30 minutes?**
b) **What is the temperature will the metal be after two hours?**
c) **When will the metal be $30\ ^oC$?**

Sol. **:**

$$T - T_S = (T_0 - T_S)e^{tk} \Rightarrow 50 = 60e^{20k} \Rightarrow k = \frac{\ln 5 - \ln 6}{20} = -0.0091$$

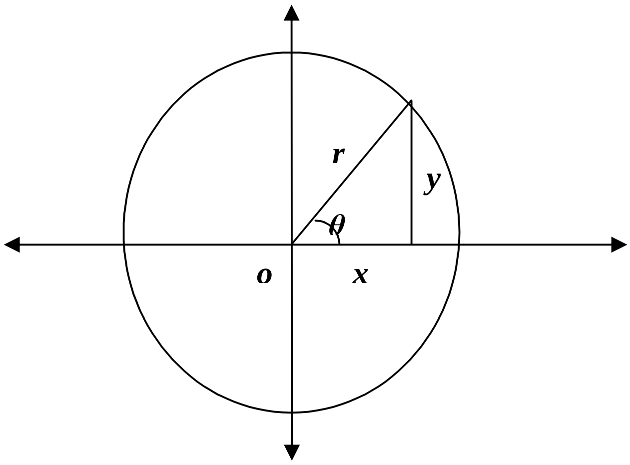
$$a)\quad T - 20 = 60e^{30(-0.0091)} = 60 * 0.761 = 45.6\ ^oC \Rightarrow T = 65.6\ ^oC$$

$$b)\quad T - T_S = 60e^{120(-0.0091)} = 60 * 0.335 = 20.1\ ^oC \Rightarrow T = 40.1\ ^oC$$

$$c)\quad 10 = 60e^{-0.0091\,t} \Rightarrow -0.0091\,t = -\ln 6 \Rightarrow t = 3.3\ hrs.$$

2-2- <u>Trigonometric functions</u> : When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :

$$Sin\theta = \frac{y}{r} = \frac{1}{csc\,\theta} \quad , \quad Cos\theta = \frac{x}{r} = \frac{1}{sec\,\theta} \quad , \quad tan\,\theta = \frac{y}{x} = \frac{1}{Cot\theta} = \frac{Sin\theta}{Cos\theta}$$



The following are some properties of these functions :

1) $Sin^2\theta + Cos^2\theta = 1$

2) $1 + tan^2\,\theta = sec^2\,\theta \quad and \quad 1 + Cot^2\theta = csc^2\,\theta$

3) $Sin(\theta \mp \beta) = Sin\theta.Cos\beta \mp Cos\theta.Sin\beta$

4) $Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\theta.Sin\beta$

5) $tan(\theta \mp \beta) = \dfrac{tan\,\theta \mp tan\,\beta}{1 \pm tan\,\theta.tan\,\beta}$

6) $Sin2\theta = 2Sin\theta.Cos\theta \;\; and \;\; Cos2\theta = Cos^2\theta - Sin^2\theta$

7) $Cos^2\theta = \dfrac{1 + Cos2\theta}{2} \;\; and \;\; Sin^2\theta = \dfrac{1 - Cos2\theta}{2}$

8) $Sin(\theta \mp \frac{\pi}{2}) = \mp Cos\theta \;\; and \;\; Cos(\theta \mp \frac{\pi}{2}) = \pm Sin\theta$

9) $Sin(-\theta) = -Sin\theta \;\; and \;\; Cos(-\theta) = Cos\theta \;\; and \;\; tan(-\theta) = -tan\,\theta$

10) $Sin\theta.Sin\beta = \frac{1}{2}[Cos(\theta - \beta) - Cos(\theta + \beta)]$

$Cos\theta.Cos\beta = \frac{1}{2}[Cos(\theta - \beta) + Cos(\theta + \beta)]$

$Sin\theta.Cos\beta = \frac{1}{2}[Sin(\theta - \beta) + Sin(\theta + \beta)]$

$$11)\quad Sin\theta + Sin\beta = 2Sin\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

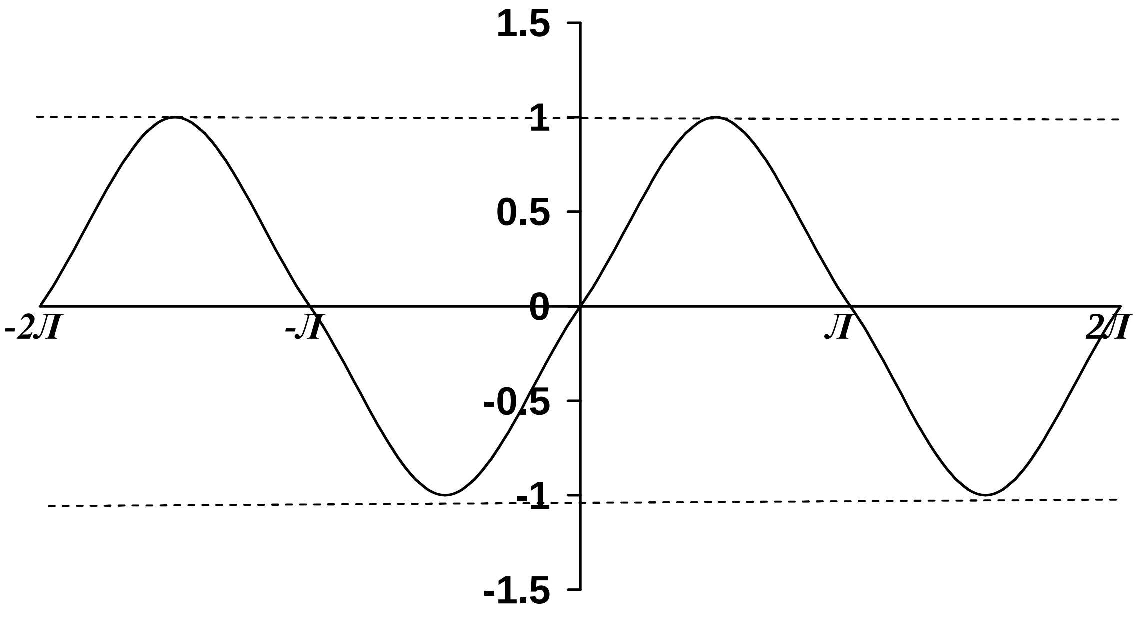
$$Sin\theta - Sin\beta = 2Cos\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

$$12)\quad Cos\theta + Cos\beta = 2Cos\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

$$Cos\theta - Cos\beta = -2Sin\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

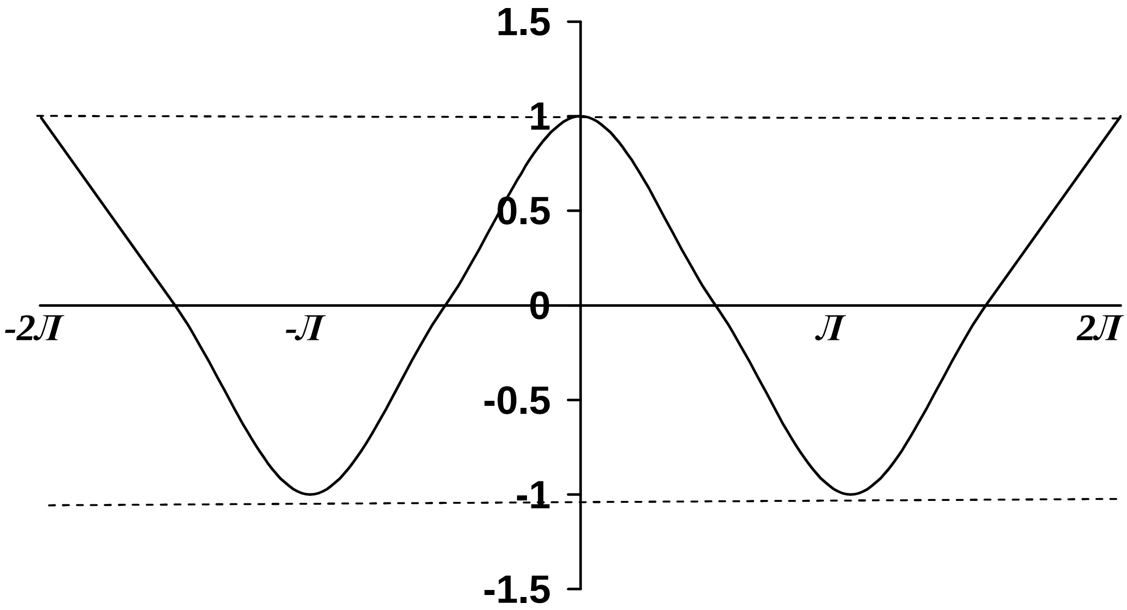
θ	0	$\Pi/6$	$\Pi/4$	$\Pi/3$	$\Pi/2$	Π
$Sin\theta$	0	1/2	1/√2	√3/2	1	0
$Cos\theta$	1	√3/2	1/√2	1/2	0	-1
$tan\theta$	0	1/√2	1	√3	∞	0

Graphs of the trigonometric functions are :

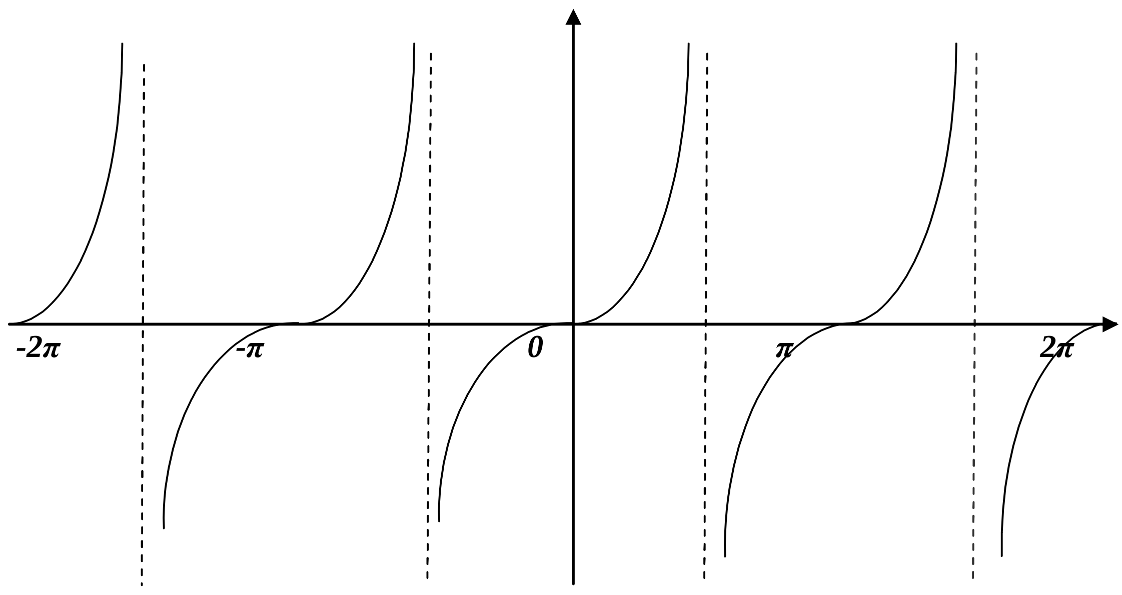


$$y = Sinx \quad D_x : \forall x$$

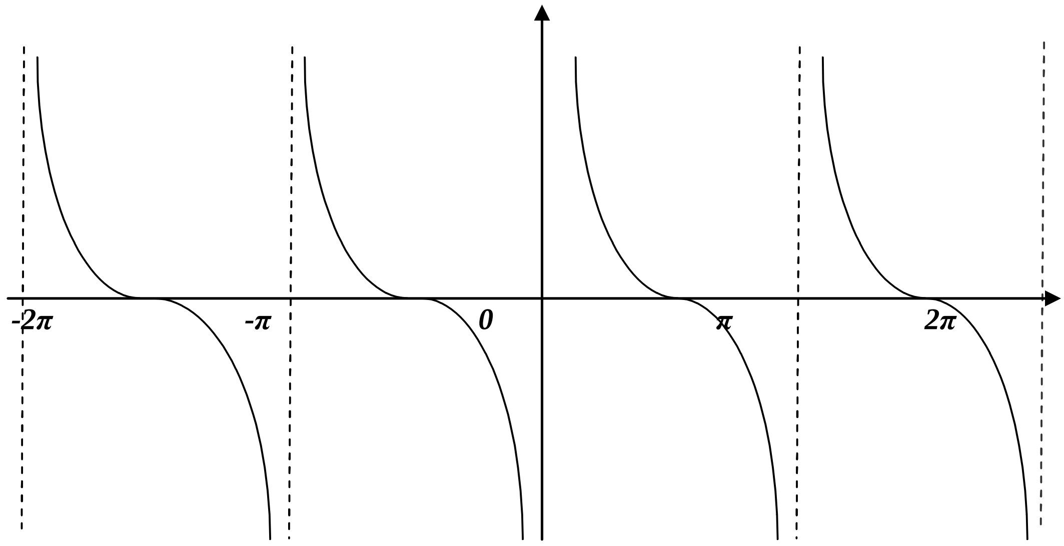
$$R_y : -1 \le y \le 1$$



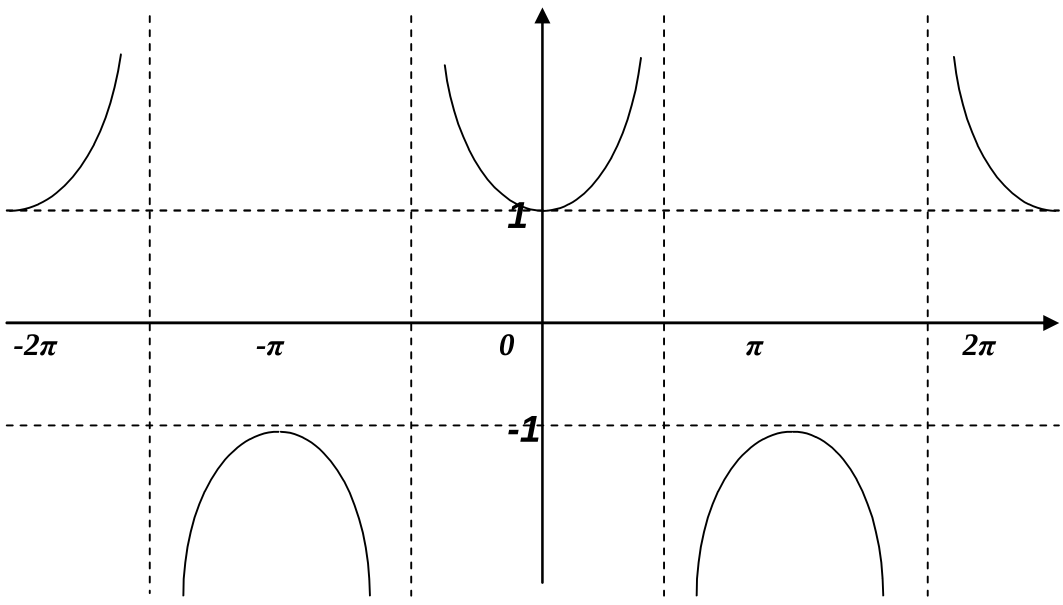
$$y = Cosx \quad D_x : \forall x$$
$$R_y : -1 \le y \le 1$$



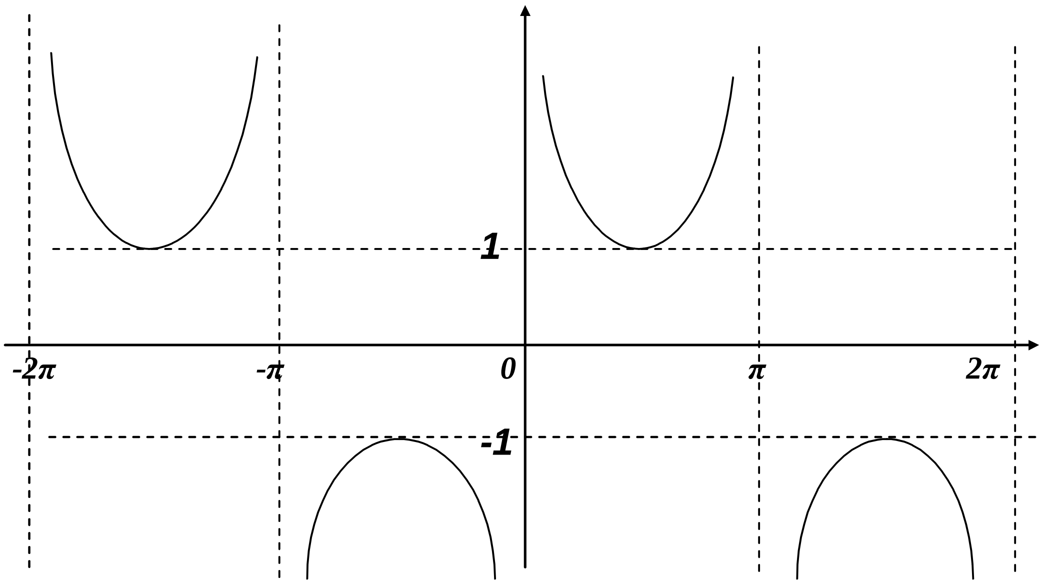
$$y = \tan x \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y$$



$$y = Cotx \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y$$



$$y = Secx \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y \geq 1 \ \text{or} \ y \leq -1$$



$$y = csc\,x \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y \geq 1 \text{ or } y \leq -1$$

$$Where \; n = 0, \mp 1, \mp 2, \mp 3, ...$$

<u>*EX-2*</u> - Solve the following equations , for values of *θ* from 0^o to 360^o inclusive .

a) $\tan\theta = 2\,Sin\,\theta$ **b)** $1 + Cos\,\theta = 2\,Sin^2\,\theta$

<u>*Sol.*</u>-

$$a)\quad tan\theta = 2Sin\theta \Rightarrow \frac{Sin\theta}{Cos\theta} = 2Sin\theta$$
$$\Rightarrow Sin\theta(1 - 2Cos\theta) = 0$$
$$either\;\; Sin\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o$$
$$or\quad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

Therefore the required values of *θ* are $0^o, 60^o, 180^o, 300^o, 360^o$.

$$b)\quad 1 + Cos\theta = 2.Sin^2\theta \Rightarrow 1 + Cos\theta = 2(1 - Cos^2\theta)$$
$$\Rightarrow (2Cos\theta - 1)(Cos\theta + 1) = 0$$
$$either\quad Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

$$or\quad Cos\theta = -1 \Rightarrow \theta = 180^o$$

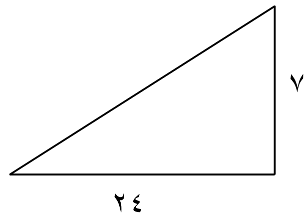
There the roots of the equation between 0^o and 360^o are $60^o, 180^o$ and 300^o .

EX-3- **If** $\tan\theta = 7/24$, **find without using tables the values of** $Sec\theta$ **and** $Sin\theta$.

Sol.-

$$\tan\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$Sec\theta = \frac{r}{x} = \frac{25}{24} \quad \text{and} \quad Sin\theta = \frac{y}{r} = \frac{7}{25}$$



EX-4- **Prove the following identities :**

$$a)\quad Csc\theta + \tan\theta . Sec\theta = Csc\theta . Sec^2\theta$$

$$b)\quad Cos^4\theta - Sin^4\theta = Cos^2\theta - Sin^2\theta$$

$$c)\quad \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta}$$

Sol.-

$$a)\quad L.H.S. = Csc\theta + \tan\theta . Sec\theta = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos\theta} . \frac{1}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Sin\theta . Cos^2\theta} = \frac{1}{Sin\theta} . \frac{1}{Cos^2\theta} = Csc\theta . Sec^2\theta = R.H.S.$$

$$b)\quad L.H.S. = Cos^4\theta - Sin^4\theta = (Cos^2\theta - Sin^2\theta).(Cos^2\theta + Sin^2\theta)$$

$$= Cos^2\theta - Sin^2\theta = R.H.S.$$

$$c)\quad L.H.S. = \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\dfrac{1}{Cos\theta} - \dfrac{1}{Sin\theta}}{\dfrac{Sin\theta}{Cos\theta} - \dfrac{Cos\theta}{Sin\theta}} = \frac{1}{Sin\theta + Cos\theta}$$

$$= \frac{Sin^2\theta + Cos^2\theta}{Sin\theta + Cos\theta} . \frac{\dfrac{1}{Sin\theta . Cos\theta}}{\dfrac{1}{Sin\theta . Cos\theta}} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta} = R.H.S.$$

EX-5- **Simplify** $\dfrac{1}{\sqrt{x^2 - a^2}}$ *when* $x = a . Csc\theta$.

Sol.- $\dfrac{1}{\sqrt{x^2 - a^2}} = \dfrac{1}{\sqrt{a^2 Csc^2\theta - a^2}} = \dfrac{1}{a\sqrt{Cot^2\theta}} = \dfrac{1}{a}\tan\theta$.

EX-6- **Eliminate** θ **from the equations :**

i) $x = a\, Sin\theta$ *and* $y = b\, \tan\theta$

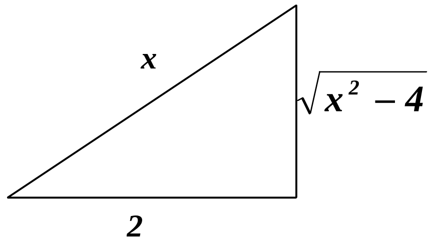
ii) $x = 2\, Sec\theta$ *and* $y = Cos2\theta$

Sol.-

$$i)\quad x = a.Sin\theta \Rightarrow Sin\theta = \frac{x}{a} \Rightarrow Csc\theta = \frac{a}{x}$$

$$y = b\tan\theta \Rightarrow \tan\theta = \frac{y}{b} \Rightarrow Cot\theta = \frac{b}{y}$$

$$Since \quad Csc^2\theta = Cot^2\theta + 1 \Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$$



$$ii)\quad x = 2\,Sec\,\theta \Rightarrow Cos\,\theta = \frac{2}{x}$$

$$y = Cos\,2\theta \Rightarrow y = Cos^2\theta - Sin^2\theta$$

$$y = \frac{4}{x^2} - \frac{x^2 - 4}{x^2} \Rightarrow x^2 y = 8 - x^2$$

EX-7- If $\tan^2\theta - 2\tan^2\beta = 1$, show that $2\,Cos^2\theta - Cos^2\beta = 0$.

Sol. –

$$\tan^2\theta - 2\tan^2\beta = 1 \Rightarrow Sec^2\theta - 1 - 2(Sec^2\beta - 1) = 1$$

$$\Rightarrow Sec^2\theta - 2Sec^2\beta = 0 \Rightarrow \frac{1}{Cos^2\theta} - \frac{2}{Cos^2\beta} = 0$$

$$\Rightarrow 2Cos^2\theta - Cos^2\beta = 0 \qquad Q.E.D.$$

EX-8- If $a\,Sin\theta = p - b\,Cos\theta$ and $b\,Sin\theta = q + a\,Cos\theta$ .Show that : $a^2 + b^2 = p^2 + q^2$

Sol.-

$$p = a.Sin\theta + b.Cos\theta \quad and \quad q = b.Sin\theta - a.Cos\theta$$

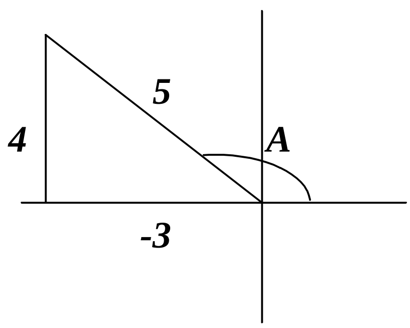
$$p^2 + q^2 = (aSin\theta + bCos\theta)^2 + (bSin\theta - aCos\theta)^2$$

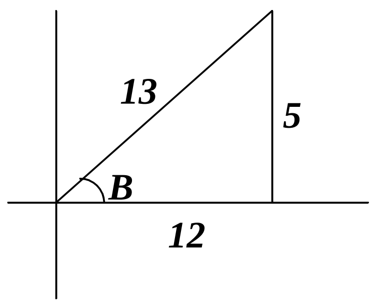
$$= a^2(Sin^2\theta + Cos^2\theta) + b^2(Cos^2\theta + Sin^2\theta) = a^2 + b^2$$

EX-9- If $Sin\,A = 4/5$ and $Cos\,B = 12/13$,where A is obtuse and B is acute . Find , without tables , the values of :

a) $Sin(A - B)$, b) $\tan(A - B)$, c) $\tan(A + B)$.

Sol. -





a) $Sin(A-B) = SinA.CosB - CosA.SinB$

$$= \frac{4}{5}.\frac{12}{13} + \frac{3}{5}.\frac{5}{13} = \frac{63}{65}$$

b) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A.\tan B}$

$$= \frac{-\frac{4}{3} - \frac{5}{12}}{1 - \frac{4}{3}.\frac{5}{12}} = -\frac{63}{16}$$

c) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A.\tan B}$

$$= \frac{-\frac{4}{3} + \frac{5}{12}}{1 + \frac{4}{3}.\frac{5}{12}} = \frac{33}{56}$$

<u>EX-10</u> **– Prove the following identities:**

a) $Sin(A+B) + Sin(A-B) = 2.SinA.CosB$

b) $\tan A + \tan B = \dfrac{Sin(A+B)}{CosA.CosB}$

c) $Sec(A+B) = \dfrac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB}$

d) $\dfrac{Sin2\theta + Cos2\theta + 1}{Sin2\theta - Cos2\theta + 1} = Cot\theta$

Sol.-

$$a)\quad L.H.S. = Sin(A+B)+Sin(A-B)$$
$$= SinA.CosB + CosA.SinB + SinA.CosB - CosA.SinB$$
$$= 2.SinA.CosB = R.H.S.$$

$$b)\quad R.H.S. = \frac{Sin(A+B)}{CosA.CosB} = \frac{SinA.CosB + CosA.SinB}{CosA.CosB}$$
$$= tan\,A + tan\,B = L.H.S.$$

$$c)\quad R.H.S = \frac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB} = \frac{\frac{1}{CosA}\cdot\frac{1}{CosB}\cdot\frac{1}{SinA}\cdot\frac{1}{SinB}}{\frac{1}{SinA}\cdot\frac{1}{SinB} - \frac{1}{CosA}\cdot\frac{1}{CosB}}$$
$$= \frac{1}{CosA.CosB - SinA.SinB} = \frac{1}{Cos(A+B)}$$
$$= Sec(A+B) = L.H.S.$$

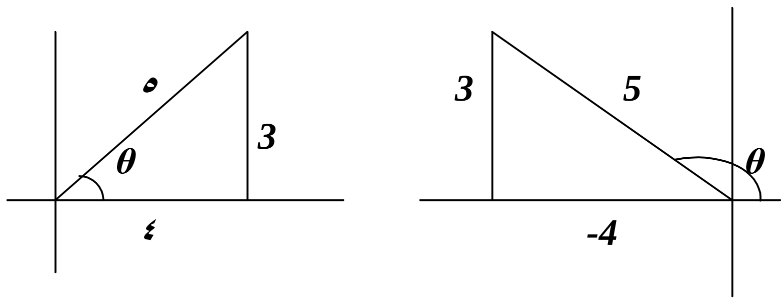
$$d)\quad L.H.S. = \frac{Sin2\theta + Cos2\theta + 1}{Sin2\theta - Cos2\theta + 1} = \frac{2Sin\theta.Cos\theta + (Cos^2\theta - Sin^2\theta) + 1}{2Sin\theta.Cos\theta - (Cos^2\theta - Sin^2\theta) + 1}$$
$$= \frac{2Sin\theta.Cos\theta + 2Cos^2\theta}{2Sin\theta.Cos\theta + 2Sin^2\theta} = \frac{Cos\theta}{Sin\theta} = Cot\theta = R.H.S.$$

EX-11 – **Find , without using tables , the values of** $Sin\,2\theta$ **and** $Cos\,2\theta$**, when:**

a) $Sin\theta = 3/5$, **b)** $Cos\,\theta = 12/13$, **c)** $Sin\,\theta = -\sqrt{3}/2$.

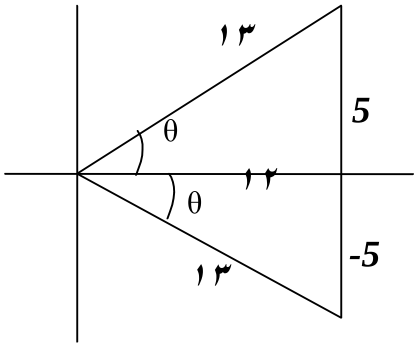
Sol. –

a)



$$Sin2\theta = 2.Sin\theta.Cos\theta = 2.\frac{3}{5}.(\mp\frac{4}{5}) = \mp\frac{24}{25}$$
$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{7}{25}$$

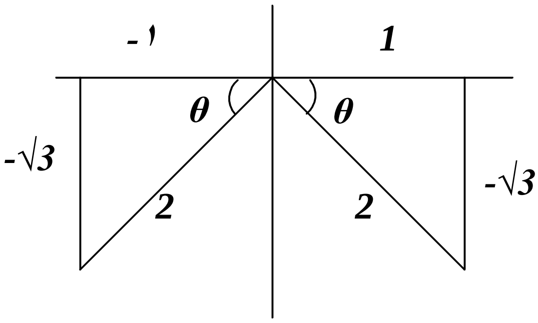
b)



$$Sin2\theta = 2.Sin\theta.Cos\theta = 2(\mp\frac{5}{13}).(\frac{12}{13}) = \mp\frac{120}{169}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\frac{12}{13})^2 - (\mp\frac{5}{13})^2 = \frac{119}{169}$$

c)



$$Sin2\theta = 2Sin\theta.Cos\theta = 2(-\frac{\sqrt{3}}{2}).(\mp\frac{1}{2}) = \pm\frac{\sqrt{3}}{2}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{1}{2})^2 - (-\frac{\sqrt{3}}{2})^2 = -\frac{1}{2}$$

EX-12- **Solve the following equations for values of θ from 0^o to 360^o inclusive:**

a) $Cos\ 2\theta + Cos\ \theta + 1 = 0$, **b)** $4\ tan\ \theta\ .\ tan\ 2\theta = 1$

***Sol.*-**

$$a)\quad Cos2\theta + Cos\theta + 1 = 0 \Rightarrow 2Cos^2\theta - 1 + Cos\theta + 1 = 0$$

$$\Rightarrow Cos(2.Cos\theta + 1) = 0$$

$$either \qquad Cos\theta = 0 \Rightarrow \theta = 90^o, 270^o$$

$$or \qquad Cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^o, 240^o$$

$$\theta = \{90^o, 120^o, 240^o, 270^o\}$$

$$b)\quad 4.\tan\theta.\tan 2\theta = 1 \Rightarrow 4.\tan\theta.\frac{2\tan\theta}{1-\tan^2\theta} = 1$$

$$\Rightarrow 9\tan^2\theta = 1$$

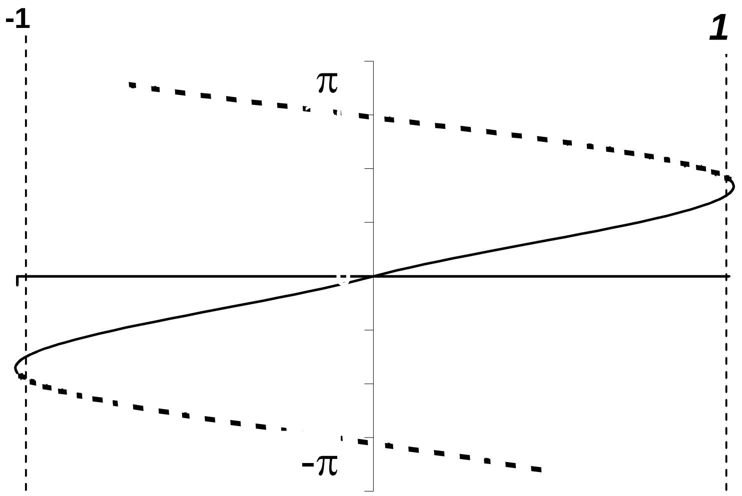
$$either \qquad \tan\theta = \frac{1}{3} \Rightarrow \theta = 18.4^o, 198.4^o$$

$$or \qquad \tan\theta = -\frac{1}{3} \Rightarrow \theta = 161.6^o, 341.6^o$$

$$\theta = \{18.4^o, 161.6^o, 198.4^o, 341.6^o\}$$

2-3- <u>The inverse trigonometric functions</u> : **The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles :**

$$y = Sinx \quad \Leftrightarrow \quad x = Sin^{-1}y$$



$$y = Sin^{-1}x \quad D_x : -1 \le x \le 1$$

$$R_y : -90 \le y \le 90$$

Chapter three
Derivatives

Let $y = f(x)$ be a function of x. If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$$

exists and is finite , we call this limit the derivative of f at x and say that f is differentiable at x .

EX-1 **– Find the derivative of the function :** $f(x) = \frac{1}{\sqrt{2x+3}}$

***Sol.*:**

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{1}{\sqrt{2(x+\Delta x)+3}} - \frac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} . \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

Rules of derivatives : Let c and n are constants, u , v and w are differentiable functions of x :

1. $\frac{d}{dx} c = 0$
2. $\frac{d}{dx} u^n = nu^{n-1} \frac{du}{dx} \Rightarrow \frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dx}$
3. $\frac{d}{dx} cu = c\frac{du}{dx}$
4. $\frac{d}{dx}(u \mp v) = \frac{du}{dx} \mp \frac{dv}{dx}$; $\frac{d}{dx}(u \mp v \mp w) = \frac{du}{dx} \mp \frac{dv}{dx} \mp \frac{dw}{dx}$
5. $\frac{d}{dx}(u.v) = u.\frac{dv}{dx} + v\frac{du}{dx}$

and $$\frac{d}{dx}(u.v.w)=u.v\frac{dw}{dx}+u.w\frac{dv}{dx}+v.w\frac{du}{dx}$$

6. $$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2} \qquad \text{where} \quad v\neq 0$$

EX-2- **Find $\frac{dy}{dx}$ for the following functions :**

$a)\ y=(x^2+1)^5$ $\qquad$ $b)\ y=[(5-x)(4-2x)]^2$

$c)\ y=(2x^3-3x^2+6x)^{-5}$ $\qquad$ $d)\ y=\frac{12}{x}-\frac{4}{x^3}+\frac{3}{x^4}$

$e)\ y=\frac{(x^2+x)(x^2-x+1)}{x^3}$ $\qquad$ $f)\ y=\frac{x^2-1}{x^2+x-2}$

Sol.-

$a)\quad \frac{dy}{dx}=5(x^2+1)^4.2x=10x(x^2+1)^4$

$b)\quad \frac{dy}{dx}=2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$\qquad =8(5-x)(2-x)(2x-7)$

$c)\quad \frac{dy}{dx}=-5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$\qquad =-30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

$d)\quad y=12x^{-1}-4x^{-3}+3x^{-4}\Rightarrow\frac{dy}{dx}=-12x^{-2}+12x^{-4}-12x^{-5}$

$\qquad \Rightarrow\frac{dy}{dx}=-\frac{12}{x^2}+\frac{12}{x^4}-\frac{12}{x^5}$

$e)\quad y=\frac{(x+1)(x^2-x+1)}{x^3}\Rightarrow$

$$\frac{dy}{dx}=\frac{x^3[(x^2-x+1)+(x+1)(2x-1)]-3x^2(x+1)(x^2-x+1)}{x^6}=-\frac{3}{x^4}$$

$f)\quad \frac{dy}{dx}=\frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2}=\frac{x^2-2x+1}{(x^2+x-2)^2}$

The Chain Rule:

1. **Suppose that $h = g_o\, f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable functions of x whose derivative at each value of x is :**

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

2. **If y is a differentiable function of t and t is differentiable function of x , then y is a differentiable function of x :**

$$y = g(t) \;\; and \;\; t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 **– Use the chain rule to express dy / dx in terms of x and y :**

$a)\quad y = \dfrac{t^2}{t^2+1} \qquad and \quad t = \sqrt{2x+1}$

$b)\quad y = \dfrac{1}{t^2+1} \qquad and \quad x = \sqrt{4t+1}$

$c)\quad y = \left(\dfrac{t-1}{t+1}\right)^2 \qquad and \quad x = \dfrac{1}{t^2} - 1 \quad at \quad t = 2$

$d)\quad y = 1 - \dfrac{1}{t} \qquad and \quad t = \dfrac{1}{1-x} \quad at \quad x = 2$

Sol.-

$$a)\quad y = \frac{t^2}{t^2+1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2+1) - 2t.t^2}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$t = (2x+1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

$$b)\quad y=(t^2+1)^{-1}\Rightarrow \frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow \frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$\text{where}\quad x=\sqrt{4t+1}\Rightarrow t=\frac{x^2-1}{4}$$

$$\text{where}\quad y=\frac{1}{t^2+1}\Rightarrow t^2+1=\frac{1}{y}$$

$$c)\quad y=\left(\frac{t-1}{t+1}\right)^2\Rightarrow \frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}=\frac{4(t-1)}{(t+1)^3}$$

$$\Rightarrow\left[\frac{dy}{dt}\right]_{t=2}=\frac{4(2-1)}{(2+1)^3}=\frac{4}{27}$$

$$x=\frac{1}{t^2}-1\Rightarrow\frac{dx}{dt}=-\frac{2}{t^3}\Rightarrow\left[\frac{dx}{dt}\right]_{t=2}=-\frac{2}{2^3}=-\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2}=\left[\frac{dy}{dt}\div\frac{dx}{dt}\right]_{t=2}=\frac{4}{27}\div\left(-\frac{1}{4}\right)=-\frac{16}{27}$$

$$d)\quad t=\frac{1}{1-x}=\frac{1}{1-2}=-1\quad \text{at}\quad x=2$$

$$y=1-\frac{1}{t}\Rightarrow\frac{dy}{dt}=\frac{1}{t^2}\Rightarrow\left[\frac{dy}{dt}\right]_{t=-1}=\frac{1}{(-1)^2}=1$$

$$t=(1-x)^{-1}\Rightarrow\frac{dt}{dx}=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

$$\Rightarrow\left[\frac{dt}{dx}\right]_{x=2}=\frac{1}{(1-2)^2}=1$$

$$\left[\frac{dy}{dx}\right]_{x=2}=\left[\frac{dy}{dt}\right]_{x=2}.\left[\frac{dt}{dx}\right]_{x=2}=1*1=1$$

<u>Higher derivatives</u> : **If a function *$y = f(x)$* possesses a derivative at every point of some interval , we may form the function *$f'(x)$* and talk**

about its derivate , if it has one . The procedure is formally identical with that used before , that is :

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$$

if the limit exists .

This derivative is called the second derivative of y with respect to x . It is written in a number of ways , for example,

$$y'' , f''(x) , or \ \frac{d^2f(x)}{dx^2} .$$

In the same manner we may define third and higher derivatives , using similar notations . The *nth* derivative may be written :

$$y^{(n)}, f^{(n)}(x), \frac{d^ny}{dx^n} .$$

***EX-4*- Find all derivatives of the following function :**

$$y = 3x^3 - 4x^2 + 7x + 10$$

***Sol.*-**

$$\frac{dy}{dx}=9x^2-8x+7 \quad , \quad \frac{d^2y}{dx^2}=18x-8$$

$$\frac{d^3y}{dx^3}=18 \quad , \quad \frac{d^4y}{dx^4}=0=\frac{d^5y}{dx^5}=....$$

***Ex-5* – Find the third derivative of the following function :**

$$y=\frac{1}{x}+\sqrt{x^3}$$

***Sol.*-**

$$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{2}{x^3}+\frac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8}x^{-\frac{3}{2}} \qquad \Rightarrow \frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8\sqrt{x^3}}$$

Implicit Differentiation: **If the formula for f is an algebraic combination of powers of x and y . To calculate the derivatives of these implicitly defined functions , we simply differentiate both sides of the defining equation with respect to x .**

***EX-6*- Find $\frac{dy}{dx}$ for the following functions:**

$a)\ x^2 . y^2 = x^2 + y^2$

$b)\ (x+y)^3 + (x-y)^3 = x^4 + y^4$

$c)\ \frac{x-y}{x-2y} = 2 \ \ at \ \ P(3,1)$

$d)\ xy + 2x - 5y = 2 \ \ at \ \ P(3,2)$

Sol.

$$a)\ x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2 y - y}$$

$$b)\ 3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c)\ \frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$$

$$d)\ x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

Exponential functions : **If u is any differentiable function of x , then :**

$$7)\ \ \frac{d}{dx}a^u = a^u . \ln a . \frac{du}{dx} \qquad and \qquad \frac{d}{dx}e^u = e^u . \frac{du}{dx}$$

EX-7 – Find $\dfrac{dy}{dx}$ for the following functions :

$a)\ y = 2^{3x}$ $\qquad$ $b)\ y = 2^x.3^x$

$c)\ y = (2^x)^2$ $\qquad$ $d)\ y = x.2^{x^2}$

$e)\ y = e^{(x+e^{5x})}$ $\qquad$ $f)\ y = e^{\sqrt{1+5x^2}}$

Sol.-

$$a)\ y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} * 3\ln 2$$

$$b)\ y = 2^x.3^x \Rightarrow y = 6^x \Rightarrow \frac{dy}{dx} = 6^x.\ln 6$$

$$c)\ y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x}\ln 2.2 = 2^{2x+1}\ln 2$$

$$d)\ y = x.2^{x^2} \Rightarrow \frac{dy}{dx} = x.2^{x^2}\ln 2.2x + 2^{x^2} = 2^{x^2}(2x^2\ln 2 + 1)$$

$$e)\ y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})}(1 + 5e^{5x})$$

$$f)\ y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}}\frac{1}{2}(1+5x^2)^{-\frac{1}{2}}.10x = e^{\sqrt{1+5x^2}}\ \frac{5x}{\sqrt{1+5x^2}}$$

Logarithm functions : If u is any differentiable function of x , then :

$$8)\ \frac{d}{dx}\log_a u = \frac{1}{u.\ln a}.\frac{du}{dx} \qquad and \qquad \frac{d}{dx}\ln u = \frac{1}{u}.\frac{du}{dx}$$

EX-8 – Find $\dfrac{dy}{dx}$ for the following functions :

$a)\ y = \log_{10} e^x$ $\qquad$ $b)\ y = \log_5(x+1)^2$

$c)\ y = \log_2(3x^2+1)^3$ $\qquad$ $d)\ y = \left[\ln(x^2+2)^2\right]^3$

$e)\ y + \ln(xy) = 1$ $\qquad$ $f)\ y = \dfrac{(2x^3-4)^{\frac{2}{3}}.(2x^2+3)^{\frac{5}{2}}}{(7x^3+4x-3)^2}$

Sol. –

$$a)\ y=\log_{10}e^{x} \Rightarrow y = x\log_{10} e \Rightarrow \frac{dy}{dx} = \log_{10} e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10}$$

$$b)\ y=\log_5 (x+1)^2 = 2\log_5 (x+1) \Rightarrow \frac{dy}{dx} = \frac{2}{(x+1)\ln 5}$$

$$c)\ y = 3\log_2 (3x^2+1) \Rightarrow \frac{dy}{dx} = \frac{3}{3x^2+1} \cdot \frac{6x}{\ln 2} = \frac{18x}{(3x^2+1)\ln 2}$$

$$d)\ \frac{dy}{dx} = 3\left[2\ln(x^2+2)\right]^2 \frac{2}{x^2+2} .2x = \frac{48x\left[\ln(x^2+2)\right]^2}{x^2+2}$$

$$e)\ y+\ln x+\ln y = 1 \Rightarrow \frac{dy}{dx} + \frac{1}{x} + \frac{1}{y} . \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x(y+1)}$$

$$f)\ \ln y = \frac{2}{3}\ln(2x^3-4) + \frac{5}{2}\ln(2x^2+3) - 2\ln(7x^3+4x-3)$$

$$\Rightarrow \frac{1}{y} . \frac{dy}{dx} = \frac{2}{3} . \frac{6x^2}{2x^3-4} + \frac{5}{2} . \frac{4x}{2x^2+3} - 2 . \frac{21x^2+4}{7x^3+4x-3}$$

$$\Rightarrow \frac{dy}{dx} = 2y\left[\frac{2x^2}{2x^3-4} + \frac{5x}{2x^2+3} - \frac{21x^2+4}{7x^3+4x-3}\right]$$

***<u>Trigonometric functions</u>* : If *u* is any differentiable function of *x* , then :**

$$9)\ \frac{d}{dx}\sin u = \cos u . \frac{du}{dx}$$

$$10)\ \frac{d}{dx}\cos u = -\sin u . \frac{du}{dx}$$

$$11)\ \frac{d}{dx}\tan u = \sec^2 u . \frac{du}{dx}$$

$$12)\ \frac{d}{dx}\cot u = -\csc^2 u . \frac{du}{dx}$$

$$13)\ \frac{d}{dx}\sec u = \sec u . \tan u . \frac{du}{dx}$$

$$14)\ \frac{d}{dx}\csc u = -\csc u . \cot u . \frac{du}{dx}$$

***<u>EX-9</u>*- Find $\frac{dy}{dx}$ for the following functions :**

$$a)\ y=\tan(3x^2) \qquad b)\ y=(\csc x+\cot x)^2$$
$$c)\ y=2\sin\frac{x}{2}-x\text{Cos}\frac{x}{2} \qquad d)\ y=\tan^2(\cos x)$$
$$e)\ x+\tan(xy)=0 \qquad f)\ y=\sec^4 x-\tan^4 x$$

Sol.-

$$a)\ \frac{dy}{dx}=\sec^2(3x^2).6x=6x.\sec^2(3x^2)$$

$$b)\ \frac{dy}{dx}=2(\csc x+\cot x)(-\csc x.\cot x-\csc^2 x)=-2\csc x.(\csc x+\cot x)^2$$

$$c)\ \frac{dy}{dx}=2\cos\frac{x}{2}.\frac{1}{2}-\left[x(-\sin\frac{x}{2}).\frac{1}{2}+\cos\frac{x}{2}\right]=\frac{x}{2}.\sin\frac{x}{2}$$

$$d)\ \frac{dy}{dx}=2.\tan(\cos x).\sec^2(\cos x).(-\sin x)=-2.\sin x.\tan(\cos x).\sec^2(\cos x)$$

$$e)\ 1+\sec^2(xy).(x\frac{dy}{dx}+y)=0\Rightarrow\frac{dy}{dx}=-\frac{1+y.\sec^2(xy)}{x.\sec^2(xy)}=-\frac{\cos^2(xy)+y}{x}$$

$$f)\ \frac{dy}{dx}=4\sec^3 x.\sec x.\tan x-4.\tan^3 x.\sec^2 x=4\tan x.\sec^2 x$$

EX-10- **Prove that :**

$$a)\ \frac{d}{dx}\tan u=\sec^2 u.\frac{du}{dx} \qquad b)\ \frac{d}{dx}\sec u=\sec u.\tan u.\frac{du}{dx}$$

Proof :

$$a)\quad L.H.S.=\frac{d}{dx}\tan u=\frac{d}{dx}\frac{\sin u}{\cos u}=\frac{\cos u.\cos u.\frac{du}{dx}-\sin u.(-\sin u)\frac{du}{dx}}{\cos^2 u}$$
$$=\frac{\cos^2 u+\sin^2 u}{\cos^2 u}.\frac{du}{dx}=\frac{1}{\cos^2 u}.\frac{du}{dx}=\sec^2 u.\frac{du}{dx}=R.H.S.$$

$$b)\quad L.H.S.=\frac{d}{dx}\sec u=\frac{d}{dx}\frac{1}{\cos u}=-\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$
$$=\frac{1}{\cos u}.\frac{\sin u}{\cos u}.\frac{du}{dx}=\sec u.\tan u.\frac{du}{dx}=R.H.S.$$

The inverse trigonometric functions : **If *u* is any differentiable function**

of x , then :

15) $$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$$

16) $$\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad -1 < u < 1$$

17) $$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

18) $$\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$$

19) $$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$$

20) $$\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \qquad |u| > 1$$

EX-11- **Find $\frac{dy}{dx}$ in each of the following functions :**

a) $y = \cot^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$ b) $y = \sin^{-1}\frac{x-1}{x+1}$

c) $y = x.\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$ d) $y = \sec^{-1}5x$

e) $y = x.\ln(\sec^{-1}x)$ f) $y = 3^{\sin^{-1}2x}$

Sol. –

a) $$\frac{dy}{dx} = -\frac{1}{1+\left(\frac{2}{x}\right)^2}2.\left(-\frac{1}{x^2}\right) + \frac{1}{1+\left(\frac{x}{2}\right)^2}.\frac{1}{2} = \frac{4}{4+x^2}$$

b) $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}.\frac{(x+1).1-(x-1).1}{(x+1)^2} = \frac{1}{(x+1)\sqrt{x}}$$

c) $$\frac{dy}{dx} = x\frac{-2}{\sqrt{1-4x^2}} + \cos^{-1}2x - \frac{1}{4}.\frac{-8x}{\sqrt{1-4x^2}} = \cos^{-1}2x$$

d) $$\frac{dy}{dx} = \frac{5}{|5x|\sqrt{25x^2-1}} = \frac{1}{|x|\sqrt{25x^2-1}}$$

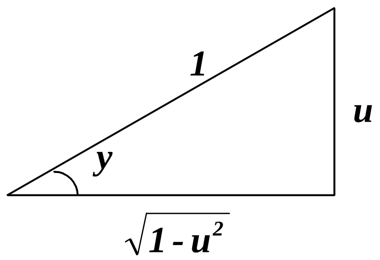
e) $\dfrac{dy}{dx}=\dfrac{x}{\sec^{-1}x}\dfrac{1}{|x|\sqrt{x^2-1}}+\ln(\sec^{-1}x)=\dfrac{1}{\sqrt{x^2-1}\,.\sec^{-1}x}+\ln(\sec^{-1}x)$

f) $\dfrac{dy}{dx}=3^{\sin^{-1}2x}.\ln 3.\dfrac{2}{\sqrt{1-4x^2}}$

<u>*EX-12*</u>- **Prove that :**

a) $\dfrac{d}{dx}\sin^{-1}u=\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ b) $\dfrac{d}{dx}\tan^{-1}u=\dfrac{1}{1+u^2}\dfrac{du}{dx}$

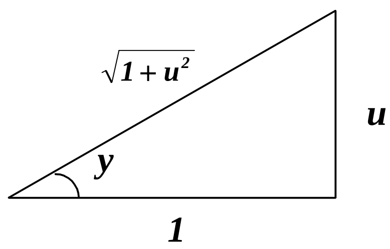
<u>*Proof*</u> *: a)*



Let $y=\sin^{-1}u\Rightarrow u=\sin y\Rightarrow \dfrac{du}{dx}=\cos y.\dfrac{dy}{dx}=\sqrt{1-u^2}\,\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}\Rightarrow \dfrac{d}{dx}\sin^{-1}u=\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$

b)



Let $y=\tan^{-1}u\Rightarrow u=\tan y\Rightarrow \dfrac{du}{dx}=\sec^2 y\,.\dfrac{du}{dx}=\left(\sqrt{1+u^2}\right)^2\dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{1}{1+u^2}.\dfrac{du}{dx}\Rightarrow \dfrac{d}{dx}\tan^{-1}u=\dfrac{1}{1+u^2}.\dfrac{du}{dx}$

Hyperbolic functions **: If *u* is any differentiable function of *x* , then :**

21) $\frac{d}{dx}\sinh u = \cosh u.\frac{du}{dx}$

22) $\frac{d}{dx}\cosh u = \sinh u.\frac{du}{dx}$

23) $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u.\frac{du}{dx}$

24) $\frac{d}{dx}\coth u = -\operatorname{csch}^2 u.\frac{du}{dx}$

25) $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u.\tanh u.\frac{du}{dx}$

26) $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u.\coth u.\frac{du}{dx}$

EX-13 **- Find $\frac{dy}{dx}$ for the following functions :**

a) $y = \coth(\tan x)$ b) $y = \sin^{-1}(\tanh x)$

c) $y = \ln\left|\tanh\frac{x}{2}\right|$ d) $y = x.\sinh 2x - \frac{1}{2}.\cosh 2x$

e) $y = \operatorname{sech}^3 x$ f) $y = \operatorname{csch}^2 x$

Sol. **-**

a) $\frac{dy}{dx} = -\operatorname{csch}^2(\tan x).\sec^2 x$

b) $\frac{dy}{dx} = \frac{\operatorname{sech}^2 x}{\sqrt{1-\tanh^2 x}} = \frac{\operatorname{sech}^2 x}{\sqrt{\operatorname{sech}^2 x}} = \operatorname{sech} x$

c) $\frac{dy}{dx} = \frac{1}{\tanh\frac{x}{2}}\operatorname{sech}^2\frac{x}{2}.\frac{1}{2} = \frac{\frac{1}{\cosh^2\frac{x}{2}}}{2.\frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}}$

$$= \frac{1}{2\sinh\frac{x}{2}.\cosh\frac{x}{2}} = \frac{1}{\sinh x} = \operatorname{csch} x$$

$$d)\ \frac{dy}{dx} = x\cosh 2x.2 + \sinh 2x - \frac{1}{2}\sinh 2x.2 = 2x\cosh 2x$$

$$e)\ \frac{dy}{dx} = 3\operatorname{sech}^2 x(-\operatorname{sech} x.\tanh x) = -3\operatorname{sech}^3 x.\tanh x$$

$$f)\ \frac{dy}{dx} = 2\operatorname{csch} x(-\operatorname{csch} x.\coth x) = -2\operatorname{csch}^2 x.\coth x$$

***<u>EX-14</u>*- Show that the functions :**

$$x = -\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} \quad and \quad y = \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \cosh\frac{t}{\sqrt{3}}$$

Taken together , satisfy the differential equations :

$$i)\ \frac{dx}{dt} + 2\frac{dy}{dt} + x = 0 \quad and \quad ii)\ \frac{dx}{dt} - \frac{dy}{dt} + y = 0$$

***<u>Proof</u>* -**

$$x = -\frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} \Rightarrow \frac{dx}{dt} = -\frac{2}{3}\cosh\frac{t}{\sqrt{3}}$$

$$y = \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \cosh\frac{t}{\sqrt{3}} \Rightarrow \frac{dy}{dt} = \frac{1}{3}\cosh\frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}}$$

$$i)\ \frac{dx}{dt} + 2\frac{dy}{dt} + x = -\frac{2}{3}\cosh\frac{t}{\sqrt{3}} + \frac{2}{3}\cosh\frac{t}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} - \frac{2}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} = 0$$

$$ii)\ \frac{dx}{dt} - \frac{dy}{dt} + y = -\frac{2}{3}\cosh\frac{t}{\sqrt{3}} - \frac{1}{3}\cosh\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \frac{1}{\sqrt{3}}\sinh\frac{t}{\sqrt{3}} + \cosh\frac{t}{\sqrt{3}} = 0$$

***<u>EX-15</u>* - Prove that :**

$$a)\ \frac{d}{dx}\tanh u = \operatorname{sech}^2 u.\frac{du}{dx} \quad and \quad b)\ \frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u.\tanh u.\frac{du}{dx}$$

***<u>Proof</u>*-**

$$a)\ \frac{d}{dx}\tanh u = \frac{d}{dx}\left(\frac{\sinh u}{\cosh u}\right) = \frac{\cosh u.\cosh u.\frac{du}{dx} - \sinh u.\sinh u.\frac{du}{dx}}{\cosh^2 u}$$

$$= \frac{(\cosh^2 u - \sinh^2 u)\frac{du}{dx}}{\cosh^2 u} = \frac{1}{\cosh^2 u}.\frac{du}{dx} = \operatorname{sech}^2 u.\frac{du}{dx}$$

$$b)\ \frac{d}{dx}\frac{1}{\cosh u} = -\frac{1}{\cosh^2 u}.\sinh u.\frac{du}{dx} = -\operatorname{sech} u.\tanh u.\frac{du}{dx}$$

The inverse hyperbolic functions **:** **If *u* is any differentiable function of *x* , then :**

27) $$\frac{d}{dx}\sinh^{-1}u=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

28) $$\frac{d}{dx}\cosh^{-1}u=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

29) $$\frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}\frac{du}{dx} \qquad |u|<1$$

30) $$\frac{d}{dx}\coth^{-1}u=\frac{1}{1-u^2}\frac{du}{dx} \qquad |u|>1$$

31) $$\frac{d}{dx}\operatorname{sech}^{-1}u=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$$

32) $$\frac{d}{dx}\operatorname{csch}^{-1}u=-\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$$

EX-16 **- Find $\frac{dy}{dx}$ for the following functions :**

a) $y=\cosh^{-1}(\sec x)$ *b)* $y=\tanh^{-1}(\cos x)$

c) $y=\coth^{-1}(\sec x)$ *d)* $y=\operatorname{sech}^{-1}(\sin 2x)$

Sol.-

a) $$\frac{dy}{dx}=\frac{\sec x.\tan x}{\sqrt{\sec^2 x-1}}=\frac{\sec x.\tan x}{\sqrt{\tan^2 x}}=\sec x \qquad \text{where } \tan x>0$$

b) $$\frac{dy}{dx}=\frac{-\sin x}{1-\cos^2 x}=\frac{-\sin x}{\sin^2 x}=-\csc x$$

c) $$\frac{dy}{dx}=\frac{\sec x.\tan x}{1-\sec^2 x}=\frac{\sec x.\tan x}{-\tan^2 x}=-\csc x$$

d) $$\frac{dy}{dx}=-\frac{2.\cos 2x}{\sin 2x.\sqrt{1-\sin^2 2x}}=-2\csc 2x \qquad \text{where } \cos 2x>0$$

EX-17 **– Verify the following formulas :**

a) $$\frac{d}{dx}\cosh^{-1}u=\frac{1}{\sqrt{u^2-1}}.\frac{du}{dx}$$

b) $$\frac{d}{dx}\tanh^{-1}u=\frac{1}{1-u^2}.\frac{du}{dx} \qquad |u|<1$$

Proof

a) Let $y=cosh^{-1}u \Rightarrow u=cosh\,y$

$$\frac{du}{dx}=sinh\,y.\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{sinh\,y}.\frac{du}{dx}$$

$$cosh^2\,y-sinh^2\,y=1\Rightarrow u^2-sinh^2\,y=1\Rightarrow sinh\,y=\sqrt{u^2-1}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{u^2-1}}.\frac{du}{dx}\Rightarrow\frac{d}{dx}cosh^{-1}u=\frac{1}{\sqrt{u^2-1}}.\frac{du}{dx}$$

b) Let $y=tanh^{-1}u \Rightarrow u=tanh\,y$

$$\frac{du}{dx}=sech^2y.\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{sech^2y}.\frac{du}{dx}$$

$$sech^2y+tanh^2\,y=1\Rightarrow sech^2y+u^2=1\Rightarrow sech^2y=1-u^2$$

$$\frac{dy}{dx}=\frac{1}{1-u^2}.\frac{du}{dx}\Rightarrow\frac{d}{dx}tanh^{-1}u=\frac{1}{1-u^2}.\frac{du}{dx}$$

The derivatives of functions like u^v **:** **Where** u **and** v **are differentiable functions of x , are found by logarithmic differentiation :**

Let $y=u^v\Rightarrow ln\,y=v.ln\,u$

$$\frac{1}{y}.\frac{dy}{dx}=\frac{v}{u}.\frac{du}{dx}+ln\,u.\frac{dv}{dx}$$

$$\frac{dy}{dx}=y\left[\frac{v}{u}.\frac{du}{dx}+ln\,u.\frac{dv}{dx}\right]$$

33) $$\frac{d}{dx}u^v=u^v.\left[\frac{v}{u}.\frac{du}{dx}+ln\,u.\frac{dv}{dx}\right]$$

EX-18- **Find** $\frac{dy}{dx}$ **for :**

a) $y=x^{cosx}$ b) $y=(lnx+x)^{tanx}$

Sol. -

a) $y=x^{cos\,x}\Rightarrow ln\,y=cos\,x.ln\,x\Rightarrow\frac{1}{y}.\frac{dy}{dx}=\frac{cos\,x}{x}+ln\,x.(-sin\,x)$

$$\Rightarrow\frac{dy}{dx}=y\left[\frac{cos\,x}{x}-sin\,x.ln\,x\right]$$

or by formula , where $u=x$ *and* $v=cosx$

$$\frac{dy}{dx}=y\left[\frac{cos\,x}{x}-sin\,x.ln\,x\right]$$

$$b)\quad y=(\ln x+x)^{\tan x}\Rightarrow \ln y=\tan x.\ln(\ln x+x)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{\tan x}{\ln x+x}.(\frac{1}{x}+1)+\ln(\ln x+x).\sec^2 x$$

$$\Rightarrow \frac{dy}{dx}=y\left[\frac{(x+1).\tan x}{x(\ln x+x)}+\ln(\ln x+x).\sec^2 x\right]$$

or by formula , where $u=\ln x+x$ *and* $v=\tan x$

$$\frac{dy}{dx}=y.\left[\frac{\tan x}{\ln x+x}(\frac{1}{x}+1)+\ln(\ln x+x).\sec^2 x\right]$$

$$=y.\left[\frac{(x+1).\tan x}{x(\ln x+x)}+\ln(\ln x+x).\sec^2 x\right]$$

Problems -3

1. Find $\frac{dy}{dx}$ for the following functions :

1) $y=(x-3)(1-x)$ $\quad(ans.: 4-2x)$

2) $y=\frac{ax+b}{x}$ $\quad(ans.: -\frac{b}{x^2})$

3) $y=\frac{3x+4}{2x+3}$ $\quad(ans.: \frac{1}{(2x+3)^2})$

4) $y=3x^3-2\sqrt{x}+\frac{5}{x^2}$ $\quad(ans.: 9x^2-\frac{1}{\sqrt{x}}-\frac{10}{x^3})$

5) $y=\left(\sqrt{x^3}-\frac{1}{\sqrt{x^3}}\right)^2$ $\quad(ans.: \frac{3(x^6-1)}{x^4})$

6) $y=(2x-1)^2(3x+2)^3+\frac{1}{(x-2)^2}$ $\quad(ans.: (2x-1)(3x+2)^2(30x-1)-\frac{2}{(x-2)^3})$

7) $y=ln(ln x)$ $\quad(ans.: \frac{1}{x.ln x})$

8) $y=ln(Cosx)$ $\quad(ans.: -tan x)$

9) $y=Sinx^3$ $\quad(ans.: 3x^2.Cosx^3)$

10) $y=Cos^{-3}(5x^2+2)$ $\quad(ans.: \frac{30x.Sin(5x^2+4)}{Cos^4(5x^2+4)})$

11) $y=tan x.sin x$ $\quad(ans.: Sinx+tan x.Secx)$

12) $y=tan(Secx)$ $\quad(ans.: Sec^2(Secx).Secx.tan x)$

13) $y=Cot^3\left(\frac{x+1}{x-1}\right)$ $\quad(ans.: \frac{6}{(x-1)^2}.Cot^2\left(\frac{x+1}{x-1}\right).Csc^2\left(\frac{x+1}{x-1}\right))$

14) $y=\frac{Cosx}{x}$ $\quad(ans.: -\frac{x.Sinx+Cosx}{x^2})$

15) $y=\sqrt{tan\sqrt{2x+7}}$ $\quad(ans.: \frac{Sec^2\sqrt{2x+7}}{2\sqrt{2x+7}\sqrt{tan\sqrt{2x+7}}})$

16) $y=x^2.Sinx$ $\quad(ans.: x^2.Cosx+2x.Sinx)$

17) $y=Csc^{-\frac{2}{3}}\sqrt{5x}$ $\quad(ans.: \frac{5}{3\sqrt{5x}}.\frac{Cot\sqrt{5x}}{Csc^{\frac{2}{3}}\sqrt{5x}})$

18) $y=x[Sin(ln x)+Cos(ln x)]$ $\quad(ans.: 2.Cos(ln x))$

19) $y = Sin^{-1}(5x^2)$ $\quad (ans.: \dfrac{10x}{\sqrt{1-25x^4}})$

20) $y = Cot^{-1}\left(\dfrac{1+x}{1-x}\right)$ $\quad (ans.: -\dfrac{1}{1+x^2})$

21) $y = tan^{-1}\sqrt{4x^3-2}$ $\quad (ans.: \dfrac{6x^2}{(4x^3-1)\sqrt{4x^3-2}})$

22) $y = Sec^{-1}(3x^2+1)^3$ $\quad (ans.: \dfrac{18x}{|3x^2+1|\sqrt{(3x^2+1)^6-1}})$

23) $y = Sin^{-1}\dfrac{x^2}{2-x} + x^2.Sec^{-1}\dfrac{x}{2}$ $\quad (ans.: \dfrac{4x-x^2}{(2-x)\sqrt{(2-x)^2-x^4}} + \dfrac{2x}{\sqrt{x^2-4}} + 2x.Sec^{-1}\dfrac{x}{2})$

24) $y = Sin^{-1}2x.Cos^{-1}2x$ $\quad (ans.: \dfrac{2(Cos^{-1}2x - Sin^{-1}2x)}{\sqrt{1-4x^2}})$

25) $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$ $\quad (ans.: \dfrac{y}{3}\left[\dfrac{1}{x} + \dfrac{1}{x+1} + \dfrac{1}{x-2} - \dfrac{2x}{x^2+1} - \dfrac{2}{2x+3}\right])$

26) $y = tan^{-1}(\ln x)$ $\quad (ans.: \dfrac{1}{x(1+(\ln x)^2)})$

27) $y^{\frac{4}{3}} = \dfrac{\sqrt{\sin x.\cos x}}{1+2.\ln x}$ $\quad (ans.: \dfrac{3y}{4}(\dfrac{\cot x}{2} - \dfrac{\tan x}{2} - \dfrac{2}{x(1+2\ln x)}))$

28) $\sqrt{y} = \dfrac{x^5.tan^{-1}x}{(3-2x).\sqrt[3]{x}}$ $\quad (ans.: 2y(\dfrac{14}{3x} + \dfrac{1}{(1+x^2).tan^{-1}x} + \dfrac{2}{3-2x}))$

29) $y = sec^{-1}e^{2x}$ $\quad (ans.: \dfrac{2}{\sqrt{e^{4x}-1}})$

30) $y = (\cos x)^{\sqrt{x}}$ $\quad (ans.: \dfrac{y}{2\sqrt{x}}(\ln\cos x - 2x.\tan x))$

31) $y = (\sin x)^{\tan x}$ $\quad (ans.: y(1+sec^2 x.\ln\sin x))$

32) $y = \sqrt{2x^2 + cosh^2(5x)}$ $\quad (ans.: \dfrac{2x + 5cosh(5x).sinh(5x)}{\sqrt{2x^2+cosh^2(5x)}})$

33) $y = sinh(\cos 2x)$ $\quad (ans.: -2\sin 2x.cosh(\cos 2x))$

34) $y = csch\dfrac{1}{x}$ $\quad (ans.: \dfrac{1}{x^2}.csch\dfrac{1}{x}.coth\dfrac{1}{x})$

35) $y = x^{2.}.tanh^2\sqrt{x}$ $\quad (ans.: x.tanh\sqrt{x}(\sqrt{x}\,sech^2\sqrt{x} + 2tanh\sqrt{x}))$

36) $y = \ln \dfrac{\sin x . \cos x + \tan^3 x}{\sqrt{x}}$ $\quad (ans.: \dfrac{\cos^2 x - \sin^2 x + 3\tan^2 x . \sec^2 x}{\sin x . \cos x + \tan^3 x} - \dfrac{1}{2x})$

37) $y = \log_4 \sin x$ $\quad (ans.: \dfrac{\cot x}{\ln 4})$

38) $y = e^{(x^2 - e^{5x})}$ $\quad (ans.: (2x - 5e^{5x})e^{(x^2 - e^{5x})})$

39) $y = e^{x^2 \tan x}$ $\quad (ans.: (x^2 \sec^2 x + 2x \tan x)e^{x^2 \tan x})$

40) $y = 7^{\csc\sqrt{2x+3}}$ $\quad (ans.: \dfrac{-7^{\csc\sqrt{2x+3}} \ln 7}{\sqrt{2x+3}} \csc\sqrt{2x+3} . \cot\sqrt{2x+3})$

41) $y = \left[\ln(x^2+2)^2\right]\cos x$ $\quad (ans.: \dfrac{4x . \cos x}{x^2+2} - 2\ln(x^2+2)\sin x)$

42) $y = \sinh^{-1}(\tan x)$ $\quad (ans.: |\sec x|)$

43) $y = \sqrt{1 + (\ln x)^2}$ $\quad (ans.: \dfrac{\ln x}{x\sqrt{1+(\ln x)^2}})$

44) $y = \dfrac{e^x}{\ln x}$ $\quad (ans.: \dfrac{e^x(x \ln x - 1)}{x(\ln x)^2})$

45) $y = x^3 \log_2(3-2x)$ $\quad (ans.: 3x^2 \log_2(3-2x) - \dfrac{2x^3}{(3-2x)\ln 2})$

46) $y = 2\cosh^{-1}\dfrac{x}{2} + \dfrac{x}{2}\sqrt{x^2-4}$ $\quad (ans.: \dfrac{x^2}{\sqrt{x^2-4}})$

2. Verify the following derivatives :

a) $\dfrac{d}{dx}\left[5x + (\sqrt{x} + \dfrac{1}{\sqrt{x}})^2\right] = 6 - \dfrac{1}{x^2}$

b) $\dfrac{d}{dx}\left[\sqrt{x}(ax^2 + bx + c)\right] = \dfrac{1}{2\sqrt{x}}(5ax^2 + 3bx + c)$

3. Find the derivative of y with respect to x in the following functions :

a) $y = \dfrac{u^2}{u^2+1}$ and $u = 3x^3 - 2$ $\quad (ans.: \dfrac{18x^2y^2}{(3x^3-2)^3})$

b) $y = \sqrt{u} + 2u$ and $u = x^2 - 3$ $\quad (ans.: \dfrac{x}{\sqrt{x^2-3}} + 4x)$

4. Find the second derivative for the following functions :

a) $y=(x+\frac{1}{x})^3$ $(ans.: 6x+\frac{6}{x^3}+\frac{12}{x^5})$

b) $f(x)=\sqrt{2x}+\frac{2\sqrt{2}}{\sqrt{x}}$ at $x=2$ $(ans.: \frac{1}{4})$

c) $x^2-2xy+y^2-16x=0$ $(ans.: \mp x^{-\frac{3}{2}})$

5. Find the third derivative of the function :

$y=\sqrt{x^3}$ $(ans.: -\frac{3}{8y})$

6. Show for $y=\frac{u}{v}$ **that** $y''=\frac{v(vu''-uv'')-2v'(vu'-uv')}{v^3}$.

7. Show for $y = u.v$ **that** $y''' = uv''' + 3u'v'' + 3u''v' + u'''v$.

8. Show that $y=35x^4-30x^2+3$ **satisfies** $(1-x^2)y''-2xy'+20y=0$.

9. Find $\frac{dy}{dx}$ **for the following implicit functions :**

a) $x^3 + 4x\sqrt{y} - \frac{5y^2}{x} = 3$ $\quad (ans.: \frac{3x^2 + 5y^2x^{-2} + 4\sqrt{y}}{10x^{-1}y - \frac{2x}{\sqrt{y}}})$

b) $\sqrt{xy} + 1 = y$ $\quad (ans.: \frac{y}{2\sqrt{xy} - x})$

c) $3xy = (x^3 + y^3)^{\frac{3}{2}}$ $\quad (ans.: \frac{3x^2\sqrt{x^3 + y^3} - 2y}{2x - 3y^2\sqrt{x^3 + y^3}})$

d) $x^3 + x.\tan^{-1} y = y$ $\quad (ans.: \frac{(1 + y^2)(3x^2 + \tan^{-1} y)}{1 + y^2 - x})$

e) $\sin^{-1}(xy) = \cos^{-1}(x - y)$ $\quad (ans.: \frac{y\sqrt{1-(x-y)^2} + \sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2} - x\sqrt{1-(x-y)^2}})$

f) $y^2.\sin(xy) = \tan x$ $\quad (ans.: \frac{\sec^2 x - y^3.\cos(xy)}{2y.\sin(xy) + xy^2.\cos(xy)})$

g) $\sinh y = \tan^2 x$ $\quad (ans.: \frac{2.\tan x.\sec^2 x}{\cosh y})$

10. Prove the following formulas :

a) $\frac{d}{dx}\cot u = -\csc^2 u.\frac{du}{dx}$

b) $\frac{d}{dx}\csc u = -\csc u.\cot u.\frac{du}{dx}$

c) $\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1 - u^2}}.\frac{du}{dx}$

d) $\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2 - 1}}.\frac{du}{dx}$

e) $\frac{d}{dx}\sinh u = \cosh u.\frac{du}{dx}$

f) $\frac{d}{dx}\csc h\, u = -\csc h\, u.\coth u.\frac{du}{dx}$

g) $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1 + u^2}}.\frac{du}{dx}$

h) $\frac{d}{dx}\sec h^{-1} u = -\frac{1}{|u|\sqrt{1 - u^2}}.\frac{du}{dx}$

11. Show that the tangent to the hyperbola $x^2 - y^2 = 1$ **at the point** *P(coshu, sinhu)* **, cuts the x-axis at the point** *(sechu , 0)* **and except when vertical , cuts the y-axis at the point** *(0 , -cschu)* **.**

Chapter four

Applications of derivatives

4-1- L'Hopital rule :

Suppose that $f(x_o) = g(x_o) = 0$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_o. Suppose also that $g'(x) \neq 0$ at every point in (a, b) except possibly x_o. Then :

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \quad \text{provided the limit exists .}$$

Differentiate f and g as long as you still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = x_o$. Stop differentiating as soon as you get something else . L'Hopital's rule does not apply when either the numerator or denominator has a finite non-zero limit .

EX-1 **– Evaluate the following limits :**

$$1)\ \lim_{x \to o} \frac{\sin x}{x} \qquad 2)\ \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$$

$$3)\ \lim_{x \to 0} \frac{x - \sin x}{x^3} \qquad 4)\ \lim_{x \to \frac{\pi}{2}} -(x - \frac{\pi}{2}).\tan x$$

Sol. **–**

$$1)\quad \lim_{x \to 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \quad using\ L'\ Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$2)\quad \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} \Rightarrow \frac{0}{0} \quad using\ L'\ Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x \to 2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

$$3)\quad \lim_{x \to 0} \frac{x - \sin x}{x^3} \Rightarrow \frac{0}{0} \quad using\ L'\ Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \Rightarrow \frac{0}{0} \quad using\ L'\ Hopital's\ rule \Rightarrow$$

$$= \frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{6}$$

$$4)\quad \lim_{x\to\frac{\pi}{2}} -(x-\frac{\pi}{2})\tan x \Rightarrow 0.\infty \text{ we can't using L'Hoptal's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{x-\frac{\pi}{2}}{\cos x}.\lim_{x\to\frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \text{ using L'Hopital's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{-\sin x}.\lim_{x\to\frac{\pi}{2}} \sin x = \frac{1}{\sin\frac{\pi}{2}}.\sin\frac{\pi}{2} = 1$$

4-2- The slope of the curve :

Secant to the curve is a line through two points on a curve.

Slopes and tangent lines :

1. **we start with what we can calculate , namely the slope of secant through *P* and a point *Q* nearby on the curve .**
2. **we find the limiting value of the secant slope (if it exists) as *Q* approaches *p* along the curve .**
3. **we take this number to be the slope of the curve at *P* and define the tangent to the curve at *P* to be the line through *p* with this slope .**

The derivative of the function *f* is the slope of the curve :

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

EX-2- **Write an equation for the tangent line at $x = 3$ of the curve :**

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

Sol.-

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2*3+3}} = \frac{1}{3}$$

The equation of the tangent line is :

$$y - \frac{1}{3} = -\frac{1}{27}(x-3) \Rightarrow 27y + x = 12$$

4-3- Velocity and acceleration and other rates of changes :

- The average velocity of a body moving along a line is :

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{displacement}{time\ travelled}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ with respect to time t .

i.e. $$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

- The rate at which the particle's velocity increase is called its acceleration a . If a particle has an initial velocity v and a constant acceleration a, then its velocity after time t is $v + at$.

$$average\ acceleration = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant , as the interval tends to zero .

i.e. $$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- The average rate of a change in a function $y = f(x)$ over the interval from x to $x + \Delta x$ is :

$$average\ rate\ of\ change = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ provided the limit exists .

EX-3- The position s (in meters) of a moving body as a function of time t (in second) is : $s = 2t^2 + 5t - 3$; find :

a) The displacement and average velocity for the time interval from $t = 0$ to $t = 2$ seconds .
b) The body's velocity at $t = 2$ seconds .

Sol.-

$a) \quad 1) \qquad \Delta s = s(t+\Delta t) - s(t) = 2(t+\Delta t)^2 + 5(t+\Delta t) - 3 - \left[2t^2 + 5t - 3\right]$

$$= (4t+5)\Delta t + 2(\Delta t)^2$$

$$at\ t = 0\ and\ \Delta t = 2 \Rightarrow \Delta s = (4*0+5)*2 + 2*2^2 = 18$$

$$2) \quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{(4t+5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t + 5 + 2.\Delta t$$

$$at\ t = 0\ \ and\ \Delta t = 2 \Rightarrow v_{av} = 4*0 + 5 + 2*2 = 9$$

$$b) \quad v(t) = \frac{d}{dt} f(t) = 4t + 5$$

$$v(2) = 4*2 + 5 = 13$$

***EX-4*-** A particle moves along a straight line so that after *t* (seconds) , its distance from *O* a fixed point on the line is *s* (meters) , where $s = t^3 - 3t^2 + 2t$:

i) when is the particle at *O* ?
ii) what is its velocity and acceleration at these times ?
iii) what is its average velocity during the first second ?
iv) what is its average acceleration between *t = 0* and *t = 2* ?

Sol. –

$$i) \quad at\ \ s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$$

$$either\ \ t = 0\ \ or\ \ t = 1\ \ or\ \ t = 2\ \ \ sec.$$

$$ii) \quad velocity = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m/s$$

$$\Rightarrow v(1) = -1m/s$$

$$\Rightarrow v(2) = 2m/s$$

$$acceleration = a(t) = 6t - 6 \Rightarrow a(0) = -6m/s^2$$

$$\Rightarrow a(1) = 0m/s^2$$

$$\Rightarrow a(2) = 6m/s^2$$

$$iii) \quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{s(1) - s(0)}{1-0} = \frac{1 - 3 + 2 - 0}{1} = 0m/s$$

$$iv) \quad a_{av} = \frac{\Delta v}{\Delta t} = \frac{v(2) - v(0)}{2-0} = \frac{2-2}{2} = 0m/s^2$$

4-4- Maxima and Minima :

***Increasing and decreasing function* :** **Let f be defined on an interval and x_1 , x_2 denoted a number on that interval :**

- If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval .

- If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval .

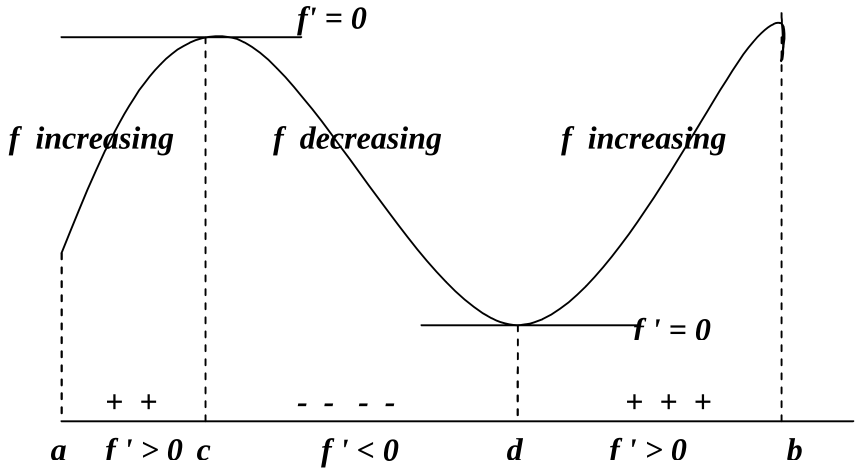
- If $f(x_1) = f(x_2)$ for all values of x_1 , x_2 then f is constant on that interval .

***The first derivative test for rise and fall* :** **Suppose that a function f has a derivative at every point x of an interval I. Then :**

- f increases on I if $f'(x) > o , \quad \forall x \in I$

- f decreases on I if $f'(x) > o , \quad \forall x \in I$

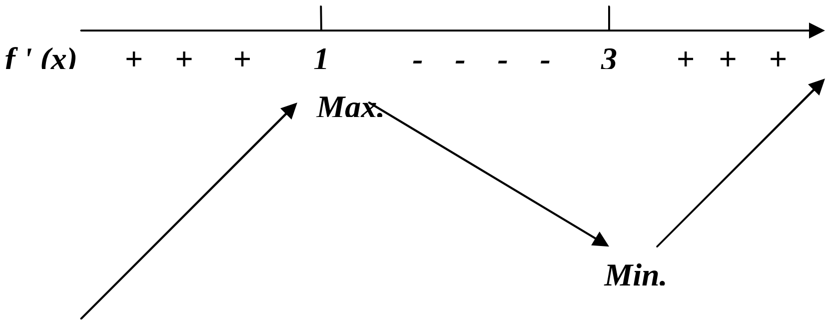
If f' changes from positive to negative values as x passes from left to right through a point c , then the value of f at c is a local maximum value of f , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.



Similarly , if f' changes from negative to positive values as x passes left to right through a point d , then the value of f at d is a local minimum value of f . That is $f(d)$ is the smallest value of f takes in the immediate neighborhood of d .

EX-5 **– Graph the function :** $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2$.

Sol.- $f'(x) = x^2 - 4x + 3 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$

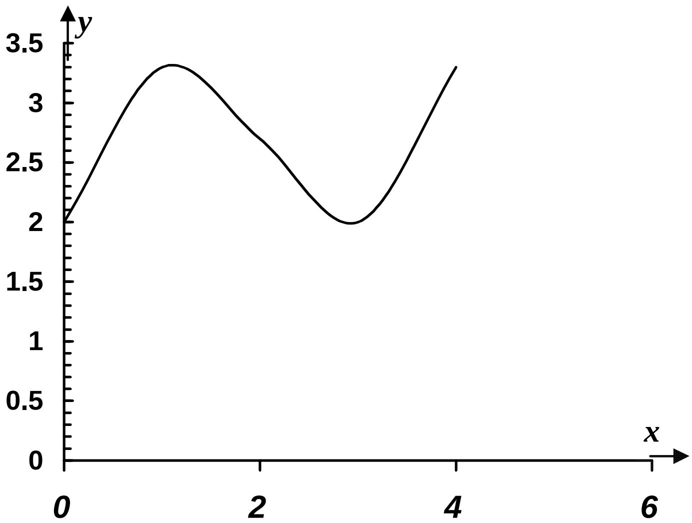


The function has a local maximum at $x = 1$ and a local minimum at $x = 3$.

To get a more accurate curve , we take :

x	*0*	*1*	*2*	*3*	*4*
f(x)	*2*	*3.3*	*2.7*	*2*	*3.3*

Then the graph of the function is :



Concave down and concave up **:** **The graph of a differentiable function $y = f(x)$ is concave down on an interval where f' decreases , and concave up on an interval where f' increases.**

The second derivative test for concavity **:** **The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, concave up on any interval where $y'' > 0$.**

Point of inflection **:** **A point on the curve where the concavity changes is called a point of inflection . Thus , a point of inflection on a twice – differentiable curve is a point where y'' is positive on one side and negative on other , i.e. $y'' = 0$.**

EX-6 – Sketch the curve : $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$.

Sol. -

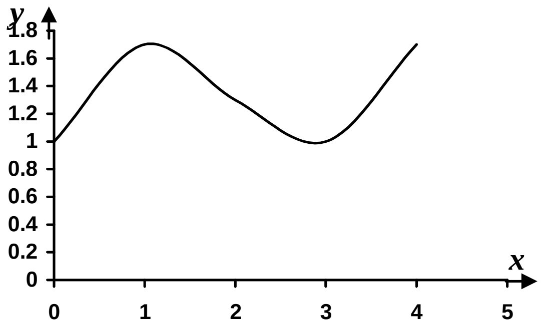
$$y' = \frac{1}{2}x^2 - 2x + \frac{3}{2} = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$$

$$y'' = x - 2 \Rightarrow at\ x = 1 \Rightarrow y'' = 1 - 2 = -1 < 0\ concave\ down\ .$$

$$\Rightarrow at\ x = 3 \Rightarrow y'' = 3 - 2 > 0 \qquad concave\ up\ .$$

$$\Rightarrow at\ y'' = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2\ \ point\ of\ inflection\ .$$

x	0	1	2	3	4
y	1	1.7	1.3	1	1.7



EX-7 – What value of a makes the function :

$$f(x) = x^2 + \frac{a}{x}\ ,\ \text{have :}$$

i) a local minimum at $x = 2$?
ii) a local minimum at $x = -3$?
iii) a point of inflection at $x = 1$?
iv) show that the function can't have a local maximum for any value of a .

Sol. –

$$f(x) = x^2 + \frac{a}{x} \Rightarrow \frac{df}{dx} = 2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3\ and\ \frac{d^2y}{dx^2} = 2 + \frac{2a}{x^3}$$

$$i)\quad at\ x=2 \Rightarrow a = 2*8 = 16\ and\ \frac{d^2f}{dx^2} = 2+\frac{2*16}{2^3} = 6 > 0\ Mini.$$

$$ii)\quad at\ x=-3 \Rightarrow a = 2(-3)^3 = -54\ and\ \frac{d^2f}{dx^2} = 2+\frac{2(-54)}{(-3)^3} = 6 > 0\ Mini.$$

$$iii)\quad at\ x=1 \Rightarrow \frac{d^2f}{dx^2} = 2+\frac{2a}{1} = 0 \Rightarrow a = -1$$

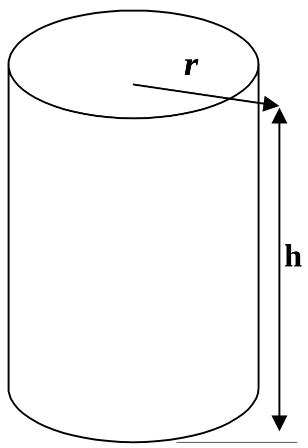
$$iv)\quad a = 2x^3 \Rightarrow \frac{d^2f}{dx^2} = 2+\frac{2(2x^3)}{x^3} = 6 > 0$$

$$Since\ \frac{d^2f}{dx^2} > 0\ \ for\ all\ value\ of\ \ x\ in\ a = 2x^3.$$

Hence the function don't have a local maximum .

<u>*EX-8*</u> – What are the best dimensions (use the least material) for a tin can which is to be in the form of a right circular cylinder and is to hold *1* gallon (*231* cubic inches) ?

<u>*Sol.*</u> – The volume of the can is :



$$v = \pi r^2 h = 231 \Rightarrow h = \frac{231}{\pi r^2}$$

where *r* is radius , *h* is height .

The total area of the outer surface (top, bottom , and side) is :

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r\frac{231}{\pi r^2} \Rightarrow A = 2\pi r^2 + \frac{462}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{462}{r^2} = 0 \Rightarrow r = 3.3252\ inches$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{924}{r^3} = 4\pi + \frac{924}{(3.3252)^3} = 37.714 > 0 \Rightarrow min.$$

$$h = \frac{231}{\pi r^2} = \frac{231}{\frac{22}{7}(3.3252)^2} = 6.6474\ inches$$

The dimensions of the can of volume *1* gallon have minimum surface area are :

***r = 3.3252* in. and *h = 6.6474* in.**

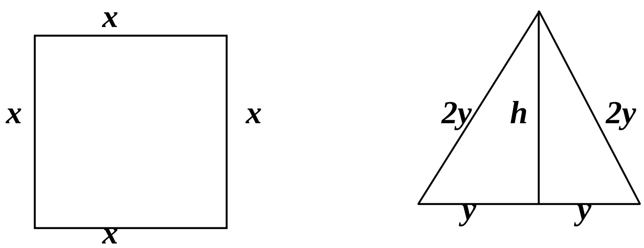
EX-9 – **A wire of length L is cut into two pieces , one being bent to form a square and the other to form an equilateral triangle . How should the wire be cut :**

a) if the sum of the two areas is minimum.

b) if the sum of the two areas is maximum.

Sol. : **Let x is a length of square.**

$2y$ is the edge of triangle .



The perimeter is $p = 4x + 6y = L \Rightarrow x = \frac{1}{4}(L - 6y)$.

$(2y)^2 = y^2 + h^2 \Rightarrow h = \sqrt{3}y$ *from triangle .*

The total area is $A = x^2 + yh = \frac{1}{16}(L - 6y)^2 + y\sqrt{3}y$

$$\Rightarrow A = \frac{1}{16}(L - 6y)^2 + \sqrt{3}y^2$$

$$\frac{dA}{dy} = \frac{-3}{4}(L - 6y) + 2\sqrt{3}y = 0 \Rightarrow y = \frac{3L}{18 + 8\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9}{2} + 2\sqrt{3} > 0 \Rightarrow min.$$

a) To minimized total areas cut for triangle $6y = \frac{9L}{9 + 4\sqrt{3}}$

And for square $L - \frac{9L}{9 + 4\sqrt{3}} = \frac{4\sqrt{3}L}{9 + 4\sqrt{3}}$.

b) To maximized the value of A on endpoints of the interval

$$0 \le 4x \le L \Rightarrow 0 \le x \le \frac{L}{4}$$

$$at\ x = 0 \Rightarrow y = \frac{L}{6}\ and\ h = \frac{L}{2\sqrt{3}} \Rightarrow A_1 = \frac{L^2}{12\sqrt{3}}$$

$$at\ \ x = \frac{L}{4} \Rightarrow y = 0 \Rightarrow A_2 = \frac{L^2}{16}$$

Since $A_2 = \frac{L^2}{16} > A_1 = \frac{L^2}{12\sqrt{3}}$

Hence the wire should not be cut at all but should be bent into a square .

Problems – 4

1. Find the velocity v if a particle's position at time t is $s = 180t – 16t^2$
When does the velocity vanish ? ***(ans.: 5.625)***

2. If a ball is thrown straight up with a velocity of 32 ft./sec. , its high after t sec. is given by the equation $s = 32t - 16t^2$. At what instant will the ball be at its highest point ? and how high will it rise ?
(ans.: 1, 16)

3. A stone is thrown vertically upwards at 35 m./sec. . Its height is : $s = 35t – 4.9t^2$ in meter above the point of projection where t is time in second later :
a) What is the distance moved, and the average velocity during the 3^{rd} sec. (from $t = 2$ to $t = 3$) ?
b) Find the average velocity for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$; $t = 2$ to $t = 2 + h$.
c) Deduce the actual velocity at the end of the 2^{nd} sec. .
(ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91, 15.4-4.9h , c) 15.4)

4. A stone is thrown vertically upwards at 24.5 m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge t sec. later is $4.9t (5 – t)$ m. . If its velocity is v m./sec. , differentiate to find v in terms of t :
i) when is the stone at the ledge level ?
ii) find its height and velocity after 1 , 2 , 3 , and 6 sec. .
iii) what meaning is attached to negative value of s ? a negative value of v ?
iv) when is the stone momentarily at rest ? what is the greatest height reached ?
v) find the total distance moved during the 3^{rd} sec. .
(ans.:v=24.5-9.8t; i)0,5; ii)19.6,29.4,29.4,-29.4;14.7,4.9, -4.9,-34.3; iv)2.5;30.625; v)2.45)

5. A stone is thrown vertically downwards with a velocity of 10 m./sec. , and gravity produces on it an acceleration of 9.8 m./sec.^{2} :
a) what is the velocity after 1 , 2 , 3 , t sec. ?
b) sketch the velocity –time graph . ***(ans.: 19.8, 29.6, 39.4,10+9.8t)***

6. A car accelerates from 5 km./h. to 41 km./h. in 10 sec. . Express this acceleration in : i)km./h. per sec. ii) m./sec.^{2}, iii) km./h.^{2} .
(ans.: i)3.6; ii)1; iii) 12960)

7. A car can accelerate at *4* m./sec.^{2} . How long will it take to reach *90* km./h. from rest ? ***(ans.: 6.25)***

8. An express train reducing its velocity to *40* km./h. , has to apply the brakes for *50* sec. . If the retardation produced is *0.5* m./sec.^{2} , find its initial velocity in km./h. . ***(ans.: 130)***

9. At the instant from which time is measured a particle is passing through *O* and traveling towards *A* , along the straight line *OA*. It is s m. from *O* after *t* sec. where $s = t(t-2)^2$:

i) when is it again at *O* ?

ii) when and where is it momentarily at rest ?

iii) what is the particle's greatest displacement from *O* , and how far does it moves , during the first *2* sec. ?

iv) what is the average velocity during the *3*rd sec. ?

v) at the end of the *1*st sec. where is the particle, which way is it going , and is its speed increasing or decreasing ?

vi) repeat (v) for the instant when $t = -1$.

(ans.:i)2;ii)0,32/27;iii)64/27;iv)3;v)OA;inceasing; vi)AO;decreasing)

10. A particle moves in a straight line so that after t sec. it is *s* m. , from a fixed point *O* on the line , where $s = t^4 + 3t^2$. Find :

i) The acceleration when $t = 1$, $t = 2$, and $t = 3$.

ii) The average acceleration between $t = 1$ and $t = 3$.

(ans.: i)18, 54,114; ii)58)

11. A particle moves along the x-axis in such away that its distance *x* cm. from the origin after *t* sec. is given by the formula $x = 27t - 2t^2$ what are its velocity and acceleration after *6.75* sec. ? How long does it take for the velocity to be reduced from *15* cm./sec. to *9* cm./sec., and how far does the particle travel mean while ? ***(ans.: 0,-4,1.5 ;18)***

12. A point moves along a straight line *OX* so that its distance *x* cm. from the point *O* at time *t* sec. is given by the formula

$x = t^3 - 6t^2 + 9t$. Find :

i) at what times and in what positions the point will have zero velocity .

ii) its acceleration at these instants .

iii) its velocity when its acceleration is zero .

(ans.: i)1,3;4,0; ii)-6,6; iii)-3)

13. A particle moves in a straight line so that its distance x cm. from a fixed point O on the line is given by $x = 9t^2 - 2t^3$ where t is the time in seconds measured from O . Find the speed of the particle when $t= 3$. Also find the distance from O of the particle when $t = 4$, and show that it is then moving towards O . *(ans.: 0, 16)*

14. Find the limits for the following functions by using L'Hopital's rule :

$1)\ \lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1}$ $\quad 2)\ \lim_{t\to 0}\frac{\sin t^2}{t}$

$3)\ \lim_{x\to\frac{\pi}{2}}\frac{2x-\pi}{\cos x}$ $\quad 4)\ \lim_{t\to 0}\frac{\cos t-1}{t^2}$

$5)\ \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x}$ $\quad 6)\ \lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$

$7)\ \lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ $\quad 8)\ \lim_{x\to 0}\frac{x(\cos x-1)}{\sin x-x}$

$9)\ \lim_{x\to 0} x.\csc^2\sqrt{2x}$ $\quad 10)\ \lim_{x\to 0}\frac{\sin x^2}{x.\sin x}$

$(ans.:1)\frac{5}{7};2)0;3)-2;4)-\frac{1}{2};5)\frac{1}{4};6)\sqrt{2};7)-1;8)3;9)\frac{1}{2};10)1)$

15. Find any local maximum and local minimum values , then sketch each curve by using first derivative :

$1)\ f(x)= x^3 - 4x^2 + 4x + 5$ $\quad (ans.: max.(0.7,6.2);min.(2,5))$

$2)\ f(x)=\frac{x^2-1}{x^2+1}$ $\quad (ans.: min.(0,-1))$

$3)\ f(x)= x^5-5x-6$ $\quad (ans.: max.(-1,-2);min.(1,-10))$

$4)\ f(x)= x^{\frac{4}{3}}-x^{\frac{1}{3}}$ $\quad (ans.: min.(0.25,-0.47))$

16. Find the interval of x-values on which the curve is concave up and concave down , then sketch the curve :

$1)\ f(x)=\frac{x^3}{3}+x^2-3x$ $\quad (ans.: up(-1,\infty);down(-\infty,-1))$

$2)\ f(x)= x^2-5x+6$ $\quad (ans.: up(-\infty,\infty))$

$3)\ f(x)= x^3 - 2x^2+1$ $\quad (ans.: up(\frac{2}{3},\infty);down(-\infty,\frac{2}{3}))$

$4)\ f(x)= x^4-2x^2$ $\quad (ans.: up(-\infty,-\frac{1}{\sqrt{3}}),(\frac{1}{\sqrt{3}},\infty);down(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}))$

17. Sketch the following curve by using second derivative :

1) $y = \dfrac{x}{1+x^2}$ *(ans.: max.(1,0.5); min.(-1,-0.5))*

2) $y = -x(x-7)^2$ *(ans.: max.(7,0); min.(2.3,-50.8))*

3) $y = (x+2)^2(x-3)$ *(ans.: max.(-2,0); min.(1.3,-18.5))*

4) $y = x^2(5-x)$ *(ans.: max.(3.3,18.5); min.(0,0))*

18. What is the smallest perimeter possible for a rectangle of area *16* in.^{2} ? *(ans.: 16)*

19. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$. *(ans.:32)*

20) A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence . With *800* m. of fence at your disposal . What is the largest area you can enclose ? *(ans.:80000)*

21) Show that the rectangle that has maximum area for a given perimeter is a square .

22) A wire of length *L* is available for making a circle and a square . How should the wire be divided between the two shapes to maximize the sum of the enclosed areas?
(ans.: all bent into a circle)

23) A closed container is made from a right circular cylinder of radius *r* and height *h* with a hemispherical dome on top . Find the relationship between *r* and *h* that maximizes the volume for a given surface area *s* . *(ans.: $r = h = \sqrt{\dfrac{s}{5\pi}}$)*

24) An open rectangular box is to be made from a piece of cardboard *8* in. wide and *15* in. long by cutting a square from each corner and bending up the sides Find the dimensions of the box of largest volume . *(ans.: height=5/3; width=14/3; length=35/3)*

Chapter five

Integration

5-1- Indefinite integrals :

The set of all anti derivatives of a function is called indefinite integral of the function.

Assume *u* and *v* denote differentiable functions of *x*, and *a*, *n*, and *c* are constants, then the integration formulas are:-

$$1)\ \int du = u(x) + c$$

$$2)\ \int a \cdot u(x)dx = a\int u(x)dx$$

$$3)\ \int \left(u(x) \mp v(x)\right)dx = \int u(x)\,dx \mp \int v(x)\,dx$$

$$4)\ \int u^n du = \frac{u^{n+1}}{n+1} + c \quad when \quad n \neq -1 \quad \& \quad \int u^{-1}du = \int \frac{1}{u}du = \ln u + c$$

$$5)\ \int a^u du = \frac{a^u}{\ln a} + c \quad \Rightarrow \quad \int e^u du = e^u + c$$

EX-1 **– Evaluate the following integrals:**

$$1)\ \int 3x^2\,dx$$

$$2)\ \int \left(\frac{1}{x^2} + x\right)dx$$

$$3)\ \int x\sqrt{x^2+1}\,dx$$

$$4)\ \int \left(2t + t^{-1}\right)^2 dt$$

$$5)\ \int \sqrt{(z^2 - z^{-2})^2 + 4}\,dz$$

$$6)\ \int \frac{x+3}{\sqrt{x^2+6x}}\,dx$$

$$7)\ \int \frac{x+2}{x^2}\,dx$$

$$8)\ \int \frac{e^x}{1+3e^x}\,dx$$

$$9)\ \int 3x^3 \cdot e^{-2x^4}\,dx$$

$$10)\ \int 2^{-4x}\,dx$$

Sol. **–**

$$1)\ \int 3x^2\,dx = 3\int x^2 dx = 3\frac{x^3}{3} + c = x^3 + c$$

$$2)\ \int (x^{-2}+x)dx = \int x^{-2}\,dx + \int x\,dx = \frac{x^{-1}}{-1} + \frac{x^2}{2} + c = -\frac{1}{x} + \frac{x^2}{2} + c$$

$$3)\ \int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int 2x(x^2+1)^{1/2}dx = \frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2} + c = \frac{1}{3}\sqrt{(x^2+1)3} + c$$

$$4)\ \int (2t+t^{-1})^2\,dt = \int (4t^2+4+t^{-2})dt = 4\frac{t^3}{3} + 4t + \frac{t^{-1}}{-1} + c = \frac{4}{3}t^3 + 4t - \frac{1}{t} + c$$

$$5)\ \int \sqrt{(z^2-z^{-2})^2+4}\,dz = \int \sqrt{z^4-2+z^{-4}+4}\,dz = \int \sqrt{z^4+2+z^{-4}}\,dz$$

$$= \int \sqrt{(z^2+z^{-2})^2}\,dz = \int (z^2+z^{-2})\,dz = \frac{z^3}{3} + \frac{z^{-1}}{-1} + c = \frac{1}{3}z^3 - \frac{1}{z} + c$$

$$6)\ \int \frac{x+3}{\sqrt{x^2+6x}}dx = \frac{1}{2}\int (2x+6)\cdot(x^2+6x)^{-1/2}dx$$

$$= \frac{1}{2}\cdot\frac{(x^2+6x)^{1/2}}{1/2} + c = \sqrt{x^2+6x} + c$$

$$7)\ \int \frac{x+2}{x^2}dx = \int\left(\frac{x}{x^2} + \frac{2}{x^2}\right)dx = \int (x^{-1}+2x^{-2})dx = \ln x + \frac{2x^{-1}}{-1} + c = \ln x - \frac{2}{x} + c$$

$$8)\ \int \frac{e^x}{1+3e^x}dx = \frac{1}{3}\int 3e^x(1+3e^x)^{-1}dx = \frac{1}{3}\ln(1+3e^x) + c$$

$$9)\ \int 3x^3\cdot e^{-2x^4}dx = -\frac{3}{8}\int -8x^3\cdot e^{-2x^4}dx = -\frac{3}{8}\cdot e^{-2x^4} + c$$

$$10)\ \int 2^{-4x}dx = -\frac{1}{4}\int 2^{-4x}\cdot(-4dx) = -\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2} + c$$

5-2- <u>Integrals of trigonometric functions</u> :

The integration formulas for the trigonometric functions are:

$$6)\ \int \sin u\cdot du = -\cos u + c \qquad 7)\ \int \cos u\cdot du = \sin u + c$$

$$8)\ \int \tan u\cdot du = -\ln|\cos u| + c \qquad 9)\ \int \cot u\cdot du = \ln|\sin u| + c$$

$$10)\ \int \sec u\cdot du = \ln|\sec u + \tan u| + c \qquad 11)\ \int \csc u\cdot du = -\ln|\csc u + \cot u| + c$$

$$12)\ \int \sec^2 u\cdot du = \tan u + c \qquad 13)\ \int \csc^2 u\cdot du = -\cot u + c$$

$$14)\ \int \sec u\cdot\tan u\cdot du = \sec u + c \qquad 15)\ \int \csc u\cdot\cot u\cdot du = -\csc u + c$$

EX-2- **Evaluate the following integrals:**

1) $\int \cos(3\theta - 1)d\theta$

2) $\int x \cdot \sin(2x^2)\,dx$

3) $\int \cos^2(2y) \cdot \sin(2y)\,dy$

4) $\int \sec^3 x \cdot \tan x\ dx$

5) $\int \sqrt{2+\sin 3t} \cdot \cos 3t\ dt$

6) $\int \frac{d\theta}{\cos^2\theta}$

7) $\int \left(1 - \sin^2 3t\right) \cdot \cos 3t\ dt$

8) $\int \tan^3(5x) \cdot \sec^2(5x)\ dx$

9) $\int \sin^4 x \cdot \cos^3 x\ dx$

10) $\int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx$

Sol.-

1) $\frac{1}{3}\int 3\cos(3\theta - 1)d\theta = \frac{1}{3}\sin(3\theta - 1) + c$

2) $\frac{1}{4}\int 4x \cdot \sin(2x^2)dx = -\frac{1}{4}\cos(2x^2) + c$

3) $-\frac{1}{2}\int (\cos 2y)^2 \cdot \left(-2\sin 2y\ dy\right) = -\frac{1}{2} \cdot \frac{(\cos 2y)^3}{3} + c = -\frac{1}{6}(\cos 2y)^3 + c$

4) $\int \sec^2 x \cdot \left(\sec x \cdot \tan x \cdot dx\right) = \frac{\sec^3 x}{3} + c$

5) $\frac{1}{3}\int (2+\sin 3t)^{1/2}\left(3\cos 3t\ dt\right) = \frac{1}{3} \cdot \frac{(2+\sin 3t)^{3/2}}{3/2} + c = \frac{2}{9}\sqrt{(2+\sin 3t)^3} + c$

6) $\int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \cdot d\theta = \tan\theta + c$

7) $\int \left(1 - \sin^2 3t\right) \cdot \cos 3t\ dt = \frac{1}{3}\int 3\cos 3t\ dt - \frac{1}{3}\int (\sin 3t)^2 \cdot 3\cos 3t\ dt$

$= \frac{1}{3}\sin 3t - \frac{1}{3} \cdot \frac{\sin^3 3t}{3} + c = \frac{1}{3} \cdot \sin 3t - \frac{1}{9}\sin^3 3t + c$

8) $\frac{1}{5}\int \tan^3 5x \cdot \left(5\sec^2 5x\ dx\right) = \frac{1}{5} \cdot \frac{\tan^4 5x}{4} + c = \frac{1}{20}\tan^4 5x + c$

9) $\int \sin^4 x \cdot \cos^3 x\ dx = \int \sin^4 x \cdot \left(1 - \sin^2 x\right) \cdot \cos x\ dx$

$= \int \sin^4 x \cdot \cos x\ dx - \int \sin^6 x \cdot \cos x\ dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

$$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2\left(-\cot\sqrt{x}\right) - \frac{x^{1/2}}{1/2} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

5-3- <u>Integrals of inverse trigonometric functions</u>:

The integration formulas for the inverse trigonometric functions are:

$$16)\ \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \qquad ; \qquad \forall u^2 < a^2$$

$$17)\ \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

$$18)\ \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \qquad ; \qquad \forall u^2 > a^2$$

<u>EX-3</u> **Evaluate the following integrals:**

$1)\ \int \frac{x^2}{\sqrt{1-x^6}}dx$

$2)\ \int \frac{dx}{\sqrt{9-x^2}}$

$3)\ \int \frac{x}{1+x^4}dx$

$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx$

$5)\ \int \frac{dx}{x\sqrt{4x^2-1}}$

$6)\ \int \frac{2dx}{\sqrt{x}(1+x)}$

$7)\ \int \frac{dx}{1+3x^2}$

$8)\ \int \frac{2\cos x}{1+\sin^2 x}dx$

$9)\ \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$10)\ \int \frac{\tan^{-1}x}{1+x^2}dx$

<u>Sol.</u>-

$$1)\ \frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}\left(3x^2\,dx\right) = \frac{1}{3}\sin^{-1}x^3 + c$$

$$2)\ \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$$

$$3)\ \frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx = \frac{1}{2}\tan^{-1} x^2 + c$$

$$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx = \sin^{-1}(\tan x) + c$$

$$5)\ \int \frac{2\ dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$$

$$6)\ \int \frac{2}{\sqrt{x}(1+x)}dx = 4\int \frac{\frac{1}{2\sqrt{x}}\ dx}{1+(\sqrt{x})^2} = 4\tan^{-1}\sqrt{x} + c$$

$$7)\ \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\ dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c$$

$$8)\ 2\int \frac{\cos x\ dx}{1+(\sin x)^2} = 2\tan^{-1}(\sin x) + c$$

$$9)\ \int e^{\sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

$$10)\ \int \tan^{-1} x \cdot \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

5-4- <u>Integrals of hyperbolic functions</u>:

The integration formulas for the hyperbolic functions are:

$$19)\ \int \sinh u \cdot du = \cosh u + c$$

$$20)\ \int \cosh u \cdot du = \sinh u + c$$

$$21)\ \int \tanh u \cdot du = \ln(\cosh u) + c$$

$$22)\ \int \coth u \cdot du = \ln(\sinh u) + c$$

$$23)\ \int \sec h^2 u \cdot du = \tanh u + c$$

$$24)\ \int \csc h^2 u \cdot du = \coth u + c$$

$$25)\ \int \sec hu \cdot \tanh u \cdot du = -\sec hu + c$$

$$26)\ \int \csc hu \cdot \coth u \cdot du = -\csc hu + c$$

EX-4 **– Evaluate the following integrals:**

1) $\int \frac{cosh(lnx)}{x} dx$

2) $\int sinh(2x+1)\,dx$

3) $\int \frac{sinhx}{cosh^4 x}\ dx$

4) $\int x\cdot cosh(3x^2)\,dx$

5) $\int sinh^4\ x\cdot cosh\,x\ \ dx$

6) $\int sec\,h^2(2x-3)\,dx$

7) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}} dx$

8) $\int \left(e^{ax}-e^{-ax}\right)dx$

9) $\int \frac{sinh\,x}{1+cosh\,x} dx$

10) $\int csch^2 x\cdot cothx\,dx$

Sol.-

1) $\int cosh(ln\,x)\cdot\left(\frac{dx}{x}\right) = sinh(ln\,x)+c$

2) $\frac{1}{2}\int sinh(2x+1)\cdot(2\,dx) = \frac{1}{2}cosh(2x+1)+c$

3) $\int \frac{1}{cosh^3\,x}\cdot\frac{sinh\,x}{cosh\,x}\ \ dx = \int sec\,h^3 x\cdot tanh\,x\ \ dx$

$= -\int sec\,h^2 x\cdot\left(-sec\,hx\cdot tanh\,x\ \ dx\right) = -\frac{sec\,h^3 x}{3}+c$

4) $\frac{1}{6}\int cosh(3x^2)\cdot(6x\,dx) = \frac{1}{6}sinh(3x^2)+c$

5) $\int sinh^4\ x\cdot\left(cosh\,x\ \ dx\right) = \frac{sinh^5\ x}{5}+c$

6) $\frac{1}{2}\int sec\,h^2\left(2x-3\right)\cdot\left(2\ \ dx\right) = \frac{1}{2}tanh\left(2x-3\right)+c$

7) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}} dx = \int tanh\,x\ \ dx = ln(cosh\,x)+c$

8) $2\int \frac{e^{ax}-e^{-ax}}{2} dx = \frac{2}{a}\int sinh\,ax\ \ (a\,dx) = \frac{2}{a}cosh\,ax+c$

9) $\int \frac{sinh\,x\ \ dx}{1+cosh\,x} = ln\left(1+cosh\,x\right)+c$

10) $-\int csc\,hx\cdot\left(-csc\,hx\cdot coth\,x\ \ dx\right) = -\frac{csc\,h^2 x}{2}+c$

5-5- Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27)\ \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28)\ \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29)\ \int \frac{du}{1-u^2} = \begin{Bmatrix} \tanh^{-1} u + c & if\ \ |u|<1 \\ \coth^{-1} u + c & if\ \ |u|>1 \end{Bmatrix} = \frac{1}{2} \ln\left|\frac{1+u}{1-u}\right| + c$$

$$30)\ \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$$

$$31)\ \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$$

EX-4 **– Evaluate the following integrals:**

$$1)\ \int \frac{dx}{\sqrt{1+4x^2}} \qquad 2)\ \int \frac{dx}{\sqrt{4+x^2}} \qquad 3)\ \int \frac{dx}{1-x^2}$$

$$4)\ \int \frac{dx}{x\sqrt{4+x^2}} \qquad 5)\ \int \frac{\sec^2\theta\ d\theta}{\sqrt{\tan^2\theta-1}} \qquad 6)\ \int \tanh^{-1}\left(\ln\sqrt{x}\right)\cdot\frac{dx}{x\left(1-\ln^2\sqrt{x}\right)}$$

Sol.-

$$1)\ \frac{1}{2}\int \frac{2\ dx}{\sqrt{1+4x^2}} = \frac{1}{2}\sinh^{-1} 2x + c$$

$$2)\ \int \frac{\frac{1}{2}\ dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1}\frac{x}{2} + c$$

$$3)\ \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \qquad if \quad |x|<1$$

$$= \coth^{-1} x + c \qquad if \quad |x|>1$$

$$4)\ \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2}\int \frac{\frac{1}{2}\ dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2}\operatorname{csc}h^{-1}\left|\frac{x}{2}\right| + c$$

$$5)\ \int \frac{1}{\sqrt{\tan^2\theta - 1}}\left(\sec^2\theta\ \ d\theta\right) = \cosh^{-1}(\tan\theta) + c$$

$$6)\quad let \quad u = \ln\sqrt{x} = \frac{1}{2}\ln x \qquad\qquad du = \frac{1}{2x}dx$$

$$\int \tanh^{-1}(\ln\sqrt{x})\cdot\frac{dx}{x(1-\ln^2\sqrt{x})} = \int \tanh^{-1}u\cdot\frac{2\,du}{1-u^2}$$

$$= 2\,\frac{\left(\tanh^{-1}u\right)^2}{2} + c = \left[\tanh^{-1}(\ln\sqrt{x})\right]^2 + c$$

Problems – 5

Evaluate the following integrals:

1) $\int \left(x^2-1\right)\cdot\left(4-x^2\right)dx$ (ans.: $\frac{5}{3}x^3-\frac{1}{5}x^5-4x+c$)

2) $\int e^x \cdot \sin e^x\, dx$ (ans.: $-\cos e^x + c$)

3) $\int \tan(3x+5)\, dx$ (ans.: $-\frac{1}{3}\ln\left|\cos(3x+5)\right| + c$)

4) $\int \frac{\cot(\ln x)}{x}dx$ (ans.: $\ln\left|\sin(\ln x)\right| + c$)

5) $\int \frac{\sin x+\cos x}{\cos x}dx$ (ans.: $-\ln\left|\cos x\right| + x + c$)

6) $\int \frac{dx}{1+\cos x}$ (ans.: $-\cot x + \csc x + c$)

7) $\int \cot(2x+1)\cdot\csc^2(2x+1)\, dx$ (ans.: $-\frac{1}{4}\cot^2(2x+1)+c$)

8) $\int \frac{dx}{\sqrt{1-9x^2}}$ (ans.: $\frac{1}{3}\sin^{-1}(3x)+c$)

9) $\int \frac{dx}{\sqrt{2-x^2}}$ (ans.: $\sin^{-1}\frac{x}{\sqrt{2}}+c$)

10) $\int e^{2x}\cdot\cosh e^{2x}\, dx$ (ans.: $\frac{1}{2}\sinh e^{2x}+c$)

11) $\int e^{\sin x}\cdot\cos x\, dx$ (ans.: $e^{\sin x}+c$)

12) $\int \frac{dx}{e^{3x}}$ (ans.: $-\frac{1}{3}e^{-3x}+c$)

13) $\int \frac{e^{\sqrt{x}}-1}{\sqrt{x}}dx$ (ans.: $2e^{\sqrt{x}}-2\sqrt{x}+c$)

14) $\int x\left(a+b\sqrt{3x}\right)dx$ where a,b constants (ans.: $\frac{1}{10}(5ax^2+4\sqrt{3}bx^{5/2})+c$)

15) $\int \frac{dx}{-1-x^2}$ (ans.: $-\tan^{-1}x+c$)

16) $\int \frac{\cos\theta\, d\theta}{1+\sin^2\theta}$ (ans.: $\tan^{-1}(\sin\theta)+c$)

17) $\int \frac{1}{x^2} \csc\frac{1}{x} \cot\frac{1}{x}\, dx$ $\quad$ (ans.: $\csc\frac{1}{x} + c$)

18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}}\, dx$ $\quad$ (ans.: $\frac{3}{4}\sqrt[3]{(3x^2+2x+1)^2} + c$)

19) $\int \sin(\tan\theta)\cdot \sec^2\theta\, d\theta$ $\quad$ (ans.: $-\cos(\tan\theta) + c$)

20) $\int \sqrt{x^2 - x^4}\, dx$ $\quad$ (ans.: $-\frac{1}{3}\sqrt{(1-x^2)^3} + c$)

21) $\int \frac{\sec^2 2x\ dx}{\sqrt{\tan 2x}}$ $\quad$ (ans.: $\sqrt{\tan 2x} + c$)

22) $\int (\sin\theta - \cos\theta)^2\, d\theta$ $\quad$ (ans.: $\theta + \cos^2\theta + c$)

23) $\int \frac{y}{y^4+1}\, dy$ $\quad$ (ans.: $\frac{1}{2}\tan^{-1} y^2 + c$)

24) $\int \frac{dx}{\sqrt{x}(x+1)}$ $\quad$ (ans.: $2\tan^{-1}\sqrt{x} + c$)

25) $\int t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}}\, dt$ $\quad$ (ans.: $\frac{9}{25}(t^{\frac{5}{3}}+1)^{\frac{5}{3}} + c$)

26) $\int \frac{dx}{x^{\frac{1}{5}}\sqrt{1+x^{\frac{4}{5}}}}$ $\quad$ (ans.: $\frac{5}{2}\sqrt{1+x^{\frac{4}{5}}} + c$)

27) $\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}}\, dx$ $\quad$ (ans.: $-\frac{1}{12}(\cos^{-1} 4x)^3 + c$)

28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ $\quad$ (ans.: $\sec^{-1}(2x) + c$)

29) $\int \frac{dx}{(e^x + e^{-x})^2}$ $\quad$ (ans.: $\frac{1}{4}\tanh x + c$)

30) $\int 3^{\ln x^2} \frac{dx}{x}$ $\quad$ (ans.: $\frac{1}{2\ln 3} 3^{\ln x^2} + c$)

31) $\int \frac{\cot x\ dx}{\ln(\sin x)}$ $\quad$ (ans.: $\ln\ln(\sin x) + c$)

32) $\int \frac{(\ln x)^2}{x}\, dx$ $\quad$ (ans.: $\frac{1}{3}(\ln x)^3 + c$)

33) $\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x}\, dx$ $\quad$ (ans.: $e^{\sec x} + c$)

34) $\int \frac{dx}{x \cdot \ln x}$ (ans.: $\ln \ln x + c$)

35) $\int \frac{d\theta}{\cosh \theta + \sinh \theta}$ (ans.: $-e^{-\theta} + c$)

36) $\int \frac{2^x - 8^{2x}}{\sqrt{4^x}} dx$ (ans.: $x - \frac{1}{5 \ln 2} 2^{5x} + c$)

37) $\int \frac{e^{\tan^{-1} 2t}}{1 + 4t^2} dt$ (ans.: $\frac{1}{2} e^{\tan^{-1} 2t} + c$)

38) $\int \frac{\cot x}{\csc x} dx$ (ans.: $\sin x + c$)

39) $\int \sec^4 x \cdot \tan^3 x \ dx$ (ans.: $\frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + c$)

40) $\int \csc^4 3x \ dx$ (ans.: $-\frac{1}{9} \cot^3 3x - \frac{1}{3} \cot 3x + c$)

41) $\int \frac{\cos^3 t}{\sin^2 t} dt$ (ans.: $-\csc t - \sin t + c$)

42) $\int \frac{\sec^4 x}{\tan^4 x} dx$ (ans.: $-\frac{1}{3} \cot^3 x - \cot x + c$)

43) $\int \tan^2 4\theta \ d\theta$ (ans.: $\frac{1}{4} \tan 4\theta - \theta + c$)

44) $\int \frac{e^x}{1 + e^x} dx$ (ans.: $\ln(1 + e^x) + c$)

45) $\int \tan^3 2x \ dx$ (ans.: $\frac{1}{4} \tan^2 2x + \frac{1}{2} \ln|\cos 2x| + c$)

46) $\int \frac{\sec^2 x}{2 + \tan x} dx$ (ans.: $\ln(2 + \tan x) + c$)

47) $\int \sec^4 3x \ dx$ (ans.: $\frac{1}{9} \tan^3 3x + \frac{1}{3} \tan 3x + c$)

48) $\int \frac{e^t}{1 + e^{2t}} dt$ (ans.: $\tan^{-1} e^t + c$)

49) $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ (ans.: $2 \sin \sqrt{x} + c$)

50) $\int \frac{dx}{\sin x \cdot \cos x}$ (ans.: $-\ln|\csc 2x + \cot 2x| + c$)

$$51)\ \int \sqrt{1+\sin y}\ dy \qquad (ans.:\ -2\sqrt{1-\sin y}+c)$$

$$52)\ \int \frac{dx}{(x^2+1)(2+\tan^{-1}x)} \qquad (ans.:\ \ln(2+\tan^{-1}x)+c)$$

$$53)\ \int \sin^{-1}(\cosh x)\cdot\frac{\sinh x\ dx}{\sqrt{1-\cosh^2 x}} \qquad (ans.:\ \frac{1}{2}\left(\sinh^{-1}(\cosh x)\right)^2+c)$$

$$54)\ \int \frac{\cos\theta\ d\theta}{1-\sin^2\theta} \qquad (ans.:\ \ln\left|\sec\theta+\tan\theta\right|+c)$$

$$55)\ \int \frac{dx}{x\left(1+(\ln x)^2\right)} \qquad (ans.:\ \tan^{-1}(\ln x)+c)$$

$$56)\ \int \left(e^{\frac{9}{4}x}-2e^{\frac{5}{4}x}+e^{\frac{x}{4}}\right)dx \qquad (ans.:\ \frac{4}{9}e^{\frac{9}{4}x}-\frac{8}{5}e^{\frac{5}{4}x}+4e^{\frac{x}{4}}+c)$$

$$57)\ \int \frac{e^x\,dx}{e^{2x}+2e^x+1} \qquad (ans.:\ -\frac{1}{e^x+1}+c)$$

$$58)\ \int e^x\cdot\sinh 2x\ dx \qquad (ans.:\ \frac{1}{2}\left[\frac{1}{3}e^{3x}+e^{-x}\right]+c)$$

$$59)\ \int \frac{\sec^3 x+e^{\sin x}}{\sec x}dx \qquad (ans.:\ \tan x+e^{\sin x}+c)$$

$$60)\ \int \frac{3^{x+2}}{2+9^{x+1}}dx \qquad (ans.:\ \frac{3}{\sqrt{2}\ln 3}\tan^{-1}\frac{3^{x+1}}{\sqrt{2}}+c)$$

$$61)\ \int \frac{\cos x\ dx}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}} \qquad (ans.:\ 2\sin^{-1}\sqrt{\sin x}+c)$$

$$62)\ \int \tan^5 x\ dx \qquad (ans.:\ \frac{1}{4}\sec^4 x-\sec^2 x-\ln\left|\cos x\right|+c)$$

$$63)\ \int e^{\ln\sin^{-1}x}\cdot\frac{dx}{\sqrt{1-x^2}} \qquad (ans.:\ \frac{1}{2}(\sin^{-1}x)^2+c)$$

$$64)\ \int x\cdot e^{x^2-1}\ dx \qquad (ans.:\ \frac{1}{2}e^{x^2-1}+c)$$

$$65)\ \int \cosh(\ln\cos x)\ dx \qquad (ans.:\ \frac{1}{2}\left[\sin x+\ln\left|\sec x+\tan x\right|\right]+c)$$

$$66)\ \int \frac{\cos x}{\sin^2 x}dx \qquad (ans.:\ -\csc x+c)$$

$$67)\ \int \cosh^{-1}(\sin x)\ \frac{\cos x\,dx}{\sqrt{\sin^2 x-1}} \qquad (ans.:\ \frac{1}{2}\left[\cosh^{-1}(\sin x)\right]^2+c)$$

Chapter five

Integration

5-1- Indefinite integrals :

The set of all anti derivatives of a function is called indefinite integral of the function.

Assume u and v denote differentiable functions of x, and a, n, and c are constants, then the integration formulas are:-

$$1)\ \int du = u(x) + c$$

$$2)\ \int a \cdot u(x)dx = a\int u(x)dx$$

$$3)\ \int \left(u(x) \mp v(x)\right)dx = \int u(x)\,dx \mp \int v(x)\,dx$$

$$4)\ \int u^n du = \frac{u^{n+1}}{n+1} + c \quad when \quad n \neq -1 \quad \& \quad \int u^{-1}du = \int \frac{1}{u}du = \ln u + c$$

$$5)\ \int a^u du = \frac{a^u}{\ln a} + c \quad \Rightarrow \quad \int e^u du = e^u + c$$

EX-1 **– Evaluate the following integrals:**

$$1)\ \int 3x^2\,dx$$

$$2)\ \int \left(\frac{1}{x^2} + x\right)dx$$

$$3)\ \int x\sqrt{x^2 + 1}\,dx$$

$$4)\ \int \left(2t + t^{-1}\right)^2 dt$$

$$5)\ \int \sqrt{(z^2 - z^{-2})^2 + 4}\,dz$$

$$6)\ \int \frac{x+3}{\sqrt{x^2 + 6x}}\,dx$$

$$7)\ \int \frac{x+2}{x^2}\,dx$$

$$8)\ \int \frac{e^x}{1 + 3e^x}\,dx$$

$$9)\ \int 3x^3 \cdot e^{-2x^4}\,dx$$

$$10)\ \int 2^{-4x}\,dx$$

Sol. **–**

$$1)\ \int 3x^2\,dx = 3\int x^2 dx = 3\frac{x^3}{3} + c = x^3 + c$$

$$2)\ \left(x^{-2}+x\right)dx=\int x^{-2}\,dx+\int x\,dx=\frac{x^{-1}}{-1}+\frac{x^2}{2}+c=-\frac{1}{x}+\frac{x^2}{2}+c$$

$$3)\ \int x\sqrt{x^2+1}\,dx=\frac{1}{2}\int 2x(x^2+1)^{1/2}dx=\frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2}+c=\frac{1}{3}\sqrt{(x^2+1)3}+c$$

$$4)\ \int\left(2t+t^{-1}\right)^2 dt=\int\left(4t^2+4+t^{-2}\right)dt=4\frac{t^3}{3}+4t+\frac{t^{-1}}{-1}+c=\frac{4}{3}t^3+4t-\frac{1}{t}+c$$

$$5)\ \int\sqrt{(z^2-z^{-2})^2+4}\,dz=\int\sqrt{z^4-2+z^{-4}+4}\,dz=\int\sqrt{z^4+2+z^{-4}}\,dz$$

$$=\int\sqrt{(z^2+z^{-2})^2}\,dz=\int(z^2+z^{-2})\,dz=\frac{z^3}{3}+\frac{z^{-1}}{-1}+c=\frac{1}{3}z^3-\frac{1}{z}+c$$

$$6)\ \int\frac{x+3}{\sqrt{x^2+6x}}dx=\frac{1}{2}\int(2x+6)\cdot(x^2+6x)^{-1/2}dx$$

$$=\frac{1}{2}\cdot\frac{(x^2+6x)^{1/2}}{1/2}+c=\sqrt{x^2+6x}+c$$

$$7)\ \int\frac{x+2}{x^2}dx=\int\left(\frac{x}{x^2}+\frac{2}{x^2}\right)dx=\int\left(x^{-1}+2x^{-2}\right)dx=\ln x+\frac{2x^{-1}}{-1}+c=\ln x-\frac{2}{x}+c$$

$$8)\ \int\frac{e^x}{1+3e^x}dx=\frac{1}{3}\int 3e^x(1+3e^x)^{-1}dx=\frac{1}{3}\ln(1+3e^x)+c$$

$$9)\ \int 3x^3\cdot e^{-2x^4}dx=-\frac{3}{8}\int -8x^3\cdot e^{-2x^4}dx=-\frac{3}{8}\cdot e^{-2x^4}+c$$

$$10)\ \int 2^{-4x}dx=-\frac{1}{4}\int 2^{-4x}\cdot(-4dx)=-\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2}+c$$

5-2- <u>Integrals of trigonometric functions</u> :

The integration formulas for the trigonometric functions are:

$$6)\ \int \sin u\cdot du=-\cos u+c$$

$$7)\ \int \cos u\cdot du=\sin u+c$$

$$8)\ \int \tan u\cdot du=-\ln|\cos u|+c$$

$$9)\ \int \cot u\cdot du=\ln|\sin u|+c$$

$$10)\ \int \sec u\cdot du=\ln|\sec u+\tan u|+c$$

$$11)\ \int \csc u\cdot du=-\ln|\csc u+\cot u|+c$$

$$12)\ \int \sec^2 u\cdot du=\tan u+c$$

$$13)\ \int \csc^2 u\cdot du=-\cot u+c$$

$$14)\ \int \sec u\cdot\tan u\cdot du=\sec u+c$$

$$15)\ \int \csc u\cdot\cot u\cdot du=-\csc u+c$$

EX-2- **Evaluate the following integrals:**

1) $\int \cos(3\theta - 1)d\theta$

2) $\int x \cdot \sin(2x^2)\,dx$

3) $\int \cos^2(2y)\cdot \sin(2y)\,dy$

4) $\int \sec^3 x \cdot \tan x\ dx$

5) $\int \sqrt{2+\sin 3t}\cdot \cos 3t\ dt$

6) $\int \frac{d\theta}{\cos^2\theta}$

7) $\int \left(1 - \sin^2 3t\right)\cdot \cos 3t\ dt$

8) $\int \tan^3(5x)\cdot \sec^2(5x)\ dx$

9) $\int \sin^4 x \cdot \cos^3 x\ dx$

10) $\int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx$

Sol.-

1) $\frac{1}{3}\int 3\cos(3\theta - 1)d\theta = \frac{1}{3}\sin(3\theta - 1) + c$

2) $\frac{1}{4}\int 4x\cdot \sin(2x^2)dx = -\frac{1}{4}\cos(2x^2) + c$

3) $-\frac{1}{2}\int (\cos 2y)^2 \cdot \left(-2\sin 2y\ dy\right) = -\frac{1}{2}\cdot\frac{(\cos 2y)^3}{3} + c = -\frac{1}{6}(\cos 2y)^3 + c$

4) $\int \sec^2 x\cdot\left(\sec x\cdot \tan x \cdot dx\right) = \frac{\sec^3 x}{3} + c$

5) $\frac{1}{3}\int (2+\sin 3t)^{1/2}\left(3\cos 3t\ dt\right) = \frac{1}{3}\cdot\frac{(2+\sin 3t)^{3/2}}{3/2} + c = \frac{2}{9}\sqrt{(2+\sin 3t)^3} + c$

6) $\int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta\cdot d\theta = \tan\theta + c$

7) $\int \left(1-\sin^2 3t\right)\cdot\cos 3t\ dt = \frac{1}{3}\int 3\cos 3t\ dt - \frac{1}{3}\int (\sin 3t)^2\cdot 3\cos 3t\ dt$

$= \frac{1}{3}\sin 3t - \frac{1}{3}\cdot\frac{\sin^3 3t}{3} + c = \frac{1}{3}\cdot\sin 3t - \frac{1}{9}\sin^3 3t + c$

8) $\frac{1}{5}\int \tan^3 5x\cdot\left(5\sec^2 5x\ dx\right) = \frac{1}{5}\cdot\frac{\tan^4 5x}{4} + c = \frac{1}{20}\tan^4 5x + c$

9) $\int \sin^4 x\cdot\cos^3 x\ dx = \int \sin^4 x\cdot\left(1-\sin^2 x\right)\cdot\cos x\ dx$

$= \int \sin^4 x\cdot\cos x\ dx - \int \sin^6 x\cdot\cos x\ dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$

$$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}\,dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2\left(-\cot\sqrt{x}\right) - \frac{x^{1/2}}{1/2} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

5-3- <u>Integrals of inverse trigonometric functions</u>:

The integration formulas for the inverse trigonometric functions are:

$$16)\ \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \qquad ; \qquad \forall u^2 < a^2$$

$$17)\ \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

$$18)\ \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \qquad ; \qquad \forall u^2 > a^2$$

<u>EX-3</u> **Evaluate the following integrals:**

$1)\ \int \frac{x^2}{\sqrt{1-x^6}}\,dx$

$2)\ \int \frac{dx}{\sqrt{9-x^2}}$

$3)\ \int \frac{x}{1+x^4}\,dx$

$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$

$5)\ \int \frac{dx}{x\sqrt{4x^2-1}}$

$6)\ \int \frac{2dx}{\sqrt{x}(1+x)}$

$7)\ \int \frac{dx}{1+3x^2}$

$8)\ \int \frac{2\cos x}{1+\sin^2 x}\,dx$

$9)\ \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$10)\ \int \frac{\tan^{-1} x}{1+x^2}\,dx$

<u>Sol.</u>-

$$1)\ \frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}\left(3x^2\,dx\right) = \frac{1}{3}\sin^{-1} x^3 + c$$

$$2)\ \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$$

$$3)\ \frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx = \frac{1}{2}\tan^{-1} x^2 + c$$

$$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx = \sin^{-1}(\tan x) + c$$

$$5)\ \int \frac{2\ dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$$

$$6)\ \int \frac{2}{\sqrt{x}(1+x)}dx = 4\int \frac{\frac{1}{2\sqrt{x}}\ dx}{1+(\sqrt{x})^2} = 4\tan^{-1}\sqrt{x} + c$$

$$7)\ \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\ dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c$$

$$8)\ 2\int \frac{\cos x\ dx}{1+(\sin x)^2} = 2\tan^{-1}(\sin x) + c$$

$$9)\ \int e^{\sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

$$10)\ \int \tan^{-1} x \cdot \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

5-4- <u>Integrals of hyperbolic functions</u>:

The integration formulas for the hyperbolic functions are:

$$19)\ \int \sinh u \cdot du = \cosh u + c$$

$$20)\ \int \cosh u \cdot du = \sinh u + c$$

$$21)\ \int \tanh u \cdot du = \ln(\cosh u) + c$$

$$22)\ \int \coth u \cdot du = \ln(\sinh u) + c$$

$$23)\ \int \sec h^2 u \cdot du = \tanh u + c$$

$$24)\ \int \csc h^2 u \cdot du = \coth u + c$$

$$25)\ \int \sec hu \cdot \tanh u \cdot du = -\sec hu + c$$

$$26)\ \int \csc hu \cdot \coth u \cdot du = -\csc hu + c$$

EX-4 **– Evaluate the following integrals:**

1) $\int \frac{cosh(lnx)}{x} dx$

2) $\int sinh(2x+1)\,dx$

3) $\int \frac{sinhx}{cosh^4 x}\,dx$

4) $\int x \cdot cosh(3x^2)\,dx$

5) $\int sinh^4 x \cdot cosh\,x\;dx$

6) $\int sec\,h^2(2x-3)\,dx$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

8) $\int \left(e^{ax} - e^{-ax}\right)dx$

9) $\int \frac{sinh\,x}{1+cosh\,x} dx$

10) $\int csch^2 x \cdot cothx\,dx$

Sol.-

1) $\int cosh(ln\,x) \cdot \left(\frac{dx}{x}\right) = sinh(ln\,x) + c$

2) $\frac{1}{2}\int sinh(2x+1)\cdot(2\,dx) = \frac{1}{2}cosh(2x+1)+c$

3) $\int \frac{1}{cosh^3 x} \cdot \frac{sinh\,x}{cosh\,x}\;dx = \int sec\,h^3 x \cdot tanh\,x\;dx$

$= -\int sec\,h^2 x \cdot \left(-sec\,hx \cdot tanh\,x\;dx\right) = -\frac{sec\,h^3 x}{3} + c$

4) $\frac{1}{6}\int cosh(3x^2)\cdot(6x\,dx) = \frac{1}{6}sinh(3x^2)+c$

5) $\int sinh^4 x \cdot \left(cosh\,x\;dx\right) = \frac{sinh^5 x}{5} + c$

6) $\frac{1}{2}\int sec\,h^2(2x-3)\cdot(2\;dx) = \frac{1}{2}tanh(2x-3)+c$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int tanh\,x\;dx = ln(cosh\,x) + c$

8) $2\int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a}\int sinh\,ax\;(a\,dx) = \frac{2}{a}cosh\,ax + c$

9) $\int \frac{sinh\,x\;dx}{1+cosh\,x} = ln\left(1+cosh\,x\right)+c$

10) $-\int csc\,hx \cdot \left(-csc\,hx \cdot coth\,x\;dx\right) = -\frac{csc\,h^2 x}{2} + c$

5-5- Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27)\ \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28)\ \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29)\ \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u|<1 \\ \coth^{-1} u + c & \text{if } |u|>1 \end{cases} = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + c$$

$$30)\ \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$$

$$31)\ \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$$

EX-4 **– Evaluate the following integrals:**

$1)\ \int \frac{dx}{\sqrt{1+4x^2}}$ $2)\ \int \frac{dx}{\sqrt{4+x^2}}$ $3)\ \int \frac{dx}{1-x^2}$

$4)\ \int \frac{dx}{x\sqrt{4+x^2}}$ $5)\ \int \frac{\sec^2\theta\ d\theta}{\sqrt{\tan^2\theta-1}}$ $6)\ \int \tanh^{-1}\left(\ln\sqrt{x}\right)\cdot\frac{dx}{x\left(1-\ln^2\sqrt{x}\right)}$

Sol.-

$$1)\ \frac{1}{2}\int \frac{2\ dx}{\sqrt{1+4x^2}} = \frac{1}{2}\sinh^{-1} 2x + c$$

$$2)\ \int \frac{\frac{1}{2}\ dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1}\frac{x}{2} + c$$

$$3)\ \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \quad \text{if } |x|<1$$

$$= \coth^{-1} x + c \quad \text{if } |x|>1$$

4) $\int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2}\int \frac{\frac{1}{2}\, dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2}\operatorname{csc}h^{-1}\left|\frac{x}{2}\right| + c$

5) $\int \frac{1}{\sqrt{\tan^2\theta - 1}}\left(\sec^2\theta \;\, d\theta\right) = \cosh^{-1}(\tan\theta) + c$

6) let $\quad u = \ln\sqrt{x} = \frac{1}{2}\ln x \qquad\qquad du = \frac{1}{2x}dx$

$$\int \tanh^{-1}(\ln\sqrt{x}\,)\cdot\frac{dx}{x(1-\ln^2\sqrt{x}\,)} = \int \tanh^{-1} u\cdot\frac{2\,du}{1-u^2}$$

$$= 2\,\frac{\left(\tanh^{-1} u\right)^2}{2} + c = \left[\tanh^{-1}(\ln\sqrt{x}\,)\right]^2 + c$$

Problems – 5

Evaluate the following integrals:

1) $\int (x^2-1)\cdot(4-x^2)\,dx$ (ans.: $\frac{5}{3}x^3-\frac{1}{5}x^5-4x+c$)

2) $\int e^x\cdot \sin e^x\,dx$ (ans.: $-\cos e^x+c$)

3) $\int \tan(3x+5)\,dx$ (ans.: $-\frac{1}{3}\ln|\cos(3x+5)|+c$)

4) $\int \frac{\cot(\ln x)}{x}\,dx$ (ans.: $\ln|\sin(\ln x)|+c$)

5) $\int \frac{\sin x+\cos x}{\cos x}\,dx$ (ans.: $-\ln|\cos x|+x+c$)

6) $\int \frac{dx}{1+\cos x}$ (ans.: $-\cot x+\csc x+c$)

7) $\int \cot(2x+1)\cdot\csc^2(2x+1)\,dx$ (ans.: $-\frac{1}{4}\cot^2(2x+1)+c$)

8) $\int \frac{dx}{\sqrt{1-9x^2}}$ (ans.: $\frac{1}{3}\sin^{-1}(3x)+c$)

9) $\int \frac{dx}{\sqrt{2-x^2}}$ (ans.: $\sin^{-1}\frac{x}{\sqrt{2}}+c$)

10) $\int e^{2x}\cdot \cosh e^{2x}\,dx$ (ans.: $\frac{1}{2}\sinh e^{2x}+c$)

11) $\int e^{\sin x}\cdot\cos x\,dx$ (ans.: $e^{\sin x}+c$)

12) $\int \frac{dx}{e^{3x}}$ (ans.: $-\frac{1}{3}e^{-3x}+c$)

13) $\int \frac{e^{\sqrt{x}}-1}{\sqrt{x}}\,dx$ (ans.: $2e^{\sqrt{x}}-2\sqrt{x}+c$)

14) $\int x\left(a+b\sqrt{3x}\right)dx$ where a,b constants (ans.: $\frac{1}{10}(5ax^2+4\sqrt{3}bx^{5/2})+c$)

15) $\int \frac{dx}{-1-x^2}$ (ans.: $-\tan^{-1}x+c$)

16) $\int \frac{\cos\theta\,d\theta}{1+\sin^2\theta}$ (ans.: $\tan^{-1}(\sin\theta)+c$)

17) $\int \frac{1}{x^2} \csc\frac{1}{x} \cot\frac{1}{x}\, dx$ $\qquad$ (ans.: $\csc\frac{1}{x} + c$)

18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}}\, dx$ $\qquad$ (ans.: $\frac{3}{4}\sqrt[3]{(3x^2+2x+1)^2} + c$)

19) $\int \sin(\tan\theta)\cdot \sec^2\theta\, d\theta$ $\qquad$ (ans.: $-\cos(\tan\theta) + c$)

20) $\int \sqrt{x^2 - x^4}\, dx$ $\qquad$ (ans.: $-\frac{1}{3}\sqrt{(1-x^2)^3} + c$)

21) $\int \frac{\sec^2 2x\ dx}{\sqrt{\tan 2x}}$ $\qquad$ (ans.: $\sqrt{\tan 2x} + c$)

22) $\int (\sin\theta - \cos\theta)^2\, d\theta$ $\qquad$ (ans.: $\theta + \cos^2\theta + c$)

23) $\int \frac{y}{y^4+1}\, dy$ $\qquad$ (ans.: $\frac{1}{2}\tan^{-1} y^2 + c$)

24) $\int \frac{dx}{\sqrt{x}(x+1)}$ $\qquad$ (ans.: $2\tan^{-1}\sqrt{x} + c$)

25) $\int t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}}\, dt$ $\qquad$ (ans.: $\frac{9}{25}(t^{\frac{5}{3}}+1)^{\frac{5}{3}} + c$)

26) $\int \frac{dx}{x^{\frac{1}{5}}\sqrt{1+x^{\frac{4}{5}}}}$ $\qquad$ (ans.: $\frac{5}{2}\sqrt{1+x^{\frac{4}{5}}} + c$)

27) $\int \frac{(\cos^{-1} 4x)^2}{\sqrt{1-16x^2}}\ dx$ $\qquad$ (ans.: $-\frac{1}{12}(\cos^{-1} 4x)^3 + c$)

28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ $\qquad$ (ans.: $\sec^{-1}(2x) + c$)

29) $\int \frac{dx}{(e^x + e^{-x})^2}$ $\qquad$ (ans.: $\frac{1}{4}\tanh x + c$)

30) $\int 3^{\ln x^2} \frac{dx}{x}$ $\qquad$ (ans.: $\frac{1}{2\ln 3} 3^{\ln x^2} + c$)

31) $\int \frac{\cot x\ dx}{\ln(\sin x)}$ $\qquad$ (ans.: $\ln\ln(\sin x) + c$)

32) $\int \frac{(\ln x)^2}{x}\, dx$ $\qquad$ (ans.: $\frac{1}{3}(\ln x)^3 + c$)

33) $\int \frac{\sin x \cdot e^{\sec x}}{\cos^2 x}\, dx$ $\qquad$ (ans.: $e^{\sec x} + c$)

$$34)\ \int \frac{dx}{x\cdot \ln x} \qquad (ans.:\ \ln \ln x + c)$$

$$35)\ \int \frac{d\theta}{\cosh\theta + \sinh\theta} \qquad (ans.:\ -e^{-\theta} + c)$$

$$36)\ \int \frac{2^x - 8^{2x}}{\sqrt{4^x}}\,dx \qquad (ans.:\ x - \frac{1}{5\ln 2}2^{5x} + c)$$

$$37)\ \int \frac{e^{\tan^{-1}2t}}{1+4t^2}\,dt \qquad (ans.:\ \frac{1}{2}e^{\tan^{-1}2t} + c)$$

$$38)\ \int \frac{\cot x}{\csc x}\,dx \qquad (ans.:\ \sin x + c)$$

$$39)\ \int \sec^4 x\cdot \tan^3 x\ dx \qquad (ans.:\ \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + c)$$

$$40)\ \int \csc^4 3x\ dx \qquad (ans.:\ -\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + c)$$

$$41)\ \int \frac{\cos^3 t}{\sin^2 t}\,dt \qquad (ans.:\ -\csc t - \sin t + c)$$

$$42)\ \int \frac{\sec^4 x}{\tan^4 x}\,dx \qquad (ans.:\ -\frac{1}{3}\cot^3 x - \cot x + c)$$

$$43)\ \int \tan^2 4\theta\ d\theta \qquad (ans.:\ \frac{1}{4}\tan 4\theta - \theta + c)$$

$$44)\ \int \frac{e^x}{1+e^x}\,dx \qquad (ans.:\ \ln(1+e^x) + c)$$

$$45)\ \int \tan^3 2x\ dx \qquad (ans.:\ \frac{1}{4}\tan^2 2x + \frac{1}{2}\ln|\cos 2x| + c)$$

$$46)\ \int \frac{\sec^2 x}{2+\tan x}\,dx \qquad (ans.:\ \ln(2+\tan x) + c)$$

$$47)\ \int \sec^4 3x\ dx \qquad (ans.:\ \frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x + c)$$

$$48)\ \int \frac{e^t}{1+e^{2t}}\,dt \qquad (ans.:\ \tan^{-1}e^t + c)$$

$$49)\ \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \qquad (ans.:\ 2\sin\sqrt{x} + c)$$

$$50)\ \int \frac{dx}{\sin x\cdot \cos x} \qquad (ans.:\ -\ln|\csc 2x + \cot 2x| + c)$$

51) $\int \sqrt{1+\sin y}\ dy$ (ans. : $-2\sqrt{1-\sin y}+c$)

52) $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$ (ans. : $\ln(2+\tan^{-1}x)+c$)

53) $\int \sin^{-1}(\cosh x)\cdot\frac{\sinh x\ dx}{\sqrt{1-\cosh^2 x}}$ (ans. : $\frac{1}{2}\left(\sinh^{-1}(\cosh x)\right)^2+c$)

54) $\int \frac{\cos\theta\ d\theta}{1-\sin^2\theta}$ (ans. : $\ln\left|\sec\theta+\tan\theta\right|+c$)

55) $\int \frac{dx}{x\left(1+(\ln x)^2\right)}$ (ans. : $\tan^{-1}(\ln x)+c$)

56) $\int \left(e^{\frac{9}{4}x}-2e^{\frac{5}{4}x}+e^{\frac{x}{4}}\right)dx$ (ans. : $\frac{4}{9}e^{\frac{9}{4}x}-\frac{8}{5}e^{\frac{5}{4}x}+4e^{\frac{x}{4}}+c$)

57) $\int \frac{e^x\,dx}{e^{2x}+2e^x+1}$ (ans. : $-\frac{1}{e^x+1}+c$)

58) $\int e^x\cdot\sinh 2x\ dx$ (ans. : $\frac{1}{2}\left[\frac{1}{3}e^{3x}+e^{-x}\right]+c$)

59) $\int \frac{\sec^3 x+e^{\sin x}}{\sec x}dx$ (ans. : $\tan x+e^{\sin x}+c$)

60) $\int \frac{3^{x+2}}{2+9^{x+1}}dx$ (ans. : $\frac{3}{\sqrt{2}\ln 3}\tan^{-1}\frac{3^{x+1}}{\sqrt{2}}+c$)

61) $\int \frac{\cos x\ dx}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}}$ (ans. : $2\sin^{-1}\sqrt{\sin x}+c$)

62) $\int \tan^5 x\ dx$ (ans. : $\frac{1}{4}\sec^4 x-\sec^2 x-\ln\left|\cos x\right|+c$)

63) $\int e^{\ln\sin^{-1}x}\cdot\frac{dx}{\sqrt{1-x^2}}$ (ans. : $\frac{1}{2}(\sin^{-1}x)^2+c$)

64) $\int x\cdot e^{x^2-1}\ dx$ (ans. : $\frac{1}{2}e^{x^2-1}+c$)

65) $\int \cosh(\ln\cos x)\ dx$ (ans. : $\frac{1}{2}\left[\sin x+\ln\left|\sec x+\tan x\right|\right]+c$)

66) $\int \frac{\cos x}{\sin^2 x}dx$ (ans. : $-\csc x+c$)

67) $\int \cosh^{-1}(\sin x)\ \frac{\cos x\ dx}{\sqrt{\sin^2 x-1}}$ (ans. : $\frac{1}{2}\left[\cosh^{-1}(\sin x)\right]^2+c$)

Chapter six

Methods of integration

6-1- Integration by parts:

The formula for integration by parts comes from the product rule:-

$$d(u\cdot v)=u\cdot dv+v\cdot du \quad \Rightarrow \quad u\cdot dv=d(u\cdot v)-v\cdot du$$

and integrated to give: $\int u\,dv=\int d(u\cdot v)-\int v\,du$

then the integration by parts formula is:-

$$\int u\,dv=u\cdot v-\int v\,du$$

Rule for choosing u **and** dv **is:**

For u**: choose something that becomes simpler when differentiated.**

For dv**: choose something whose integral is simple.**

It is not always possible to follow this rule, but when we can.

EX-1 **– Evaluate the following integrals:**

1) $\int xe^{x}\,dx$

2) $\int x\cdot cosx\,dx$

3) $\int \frac{x}{\sqrt{x-1}}\,dx$

4) $\int x^{2}\cdot ln\,x\,dx$

5) $\int x\cdot sec^{2}x\,dx$

6) $\int ln\left(x+\sqrt{1+x^{2}}\right)dx$

7) $\int sin^{-1}ax\,dx$

8) $\int e^{ax}\cdot sinbx\,dx$

9) $\int x^{3}\cdot e^{x}\,dx$

10) $\int x^{3}\cdot e^{x^{2}}\,dx$

Sol. **–**

1) let $\left.\begin{matrix} u=x \Rightarrow du=dx \\ dv=e^{x}dx \Rightarrow v=e^{x}\end{matrix}\right\} \Rightarrow \int udv=u\cdot v-\int vdu$

$$\int x\cdot e^{x}dx=x\cdot e^{x}-\int e^{x}dx=x\cdot e^{x}-e^{x}+c$$

$$2)\quad let\quad \left.\begin{aligned} &u = x \quad \Rightarrow \quad du = dx \\ &dv = cosx\, dx \ \Rightarrow\ v = sin\, x \end{aligned}\right\} \Rightarrow \quad \int u dv = u \cdot v - \int v du$$

$$\int x \cdot cosx\, dx = x \cdot sin\, x - \int sin\, x\, dx = x \cdot sin\, x + cos\, x + c$$

$$3)\quad let\quad \left.\begin{aligned} &u = x \quad \Rightarrow \quad du = dx \\ &dv = \frac{1}{\sqrt{x-1}}dx \ \Rightarrow\ v = 2(x-1)^{1/2} \end{aligned}\right\} \Rightarrow \quad \int u dv = u \cdot v - \int v du$$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x \cdot (x-1)^{1/2} - 2\int (x-1)^{1/2}dx$$

$$= 2x \cdot \sqrt{x-1} - \frac{2(x-1)^{3/2}}{3/2} + c = 2x \cdot \sqrt{x-1} - \frac{4}{3}\sqrt{(x-1)^3} + c$$

$$4)\quad let\quad \left.\begin{aligned} &u = ln\, x \quad \Rightarrow \quad du = \frac{1}{x}dx \\ &dv = x^2 dx \ \Rightarrow\ v = \frac{x^3}{3} \end{aligned}\right\} \Rightarrow \quad \int u dv = u \cdot v - \int v du$$

$$\int x^2 \cdot lnx\, dx = \frac{x^3}{3} \cdot ln\, x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3} \cdot ln\, x - \frac{1}{9}x^3 + c$$

$$5)\quad let\quad \left.\begin{aligned} &u = x \quad \Rightarrow \quad du = dx \\ &dv = sec^2 x\, dx \ \Rightarrow\ v = tan\, x \end{aligned}\right\} \Rightarrow \quad \int u dv = u \cdot v - \int v du$$

$$\int x \cdot sec^2\, x\, dx = x \cdot tan\, x - \int tan\, x\, dx = x \cdot tan\, x + ln|cos\, x| + c$$

$$6)\quad let\quad u = ln\left(x + \sqrt{1+x^2}\right) \ \Rightarrow\ du = \frac{1 + \dfrac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}dx$$

$$dv = dx \quad \Rightarrow \quad v = x$$

$$\int ln\left(x + \sqrt{1+x^2}\right)dx = x \cdot ln\left(x + \sqrt{1+x^2}\right) - \int x\left(1+x^2\right)^{-1/2}dx$$

$$= x \cdot ln\left(x + \sqrt{1+x^2}\right) - \frac{1}{2} \cdot \frac{\left(1+x^2\right)^{1/2}}{1/2} + c = x \cdot ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + c$$

7) let $u=\sin^{-1}ax \Rightarrow du=\dfrac{a\,dx}{\sqrt{1-a^2x^2}}$ & $dv=dx \Rightarrow v=x$

$$\int \sin^{-1}ax\,dx = x\cdot\sin^{-1}ax-\int\frac{a\,x}{\sqrt{1-a^2x^2}}dx$$

$$= x\cdot\sin^{-1}ax+\frac{1}{2a}\int -2a^2x\left(1-a^2x^2\right)^{-1/2}dx$$

$$= x\cdot\sin^{-1}ax+\frac{1}{2a}\cdot\frac{\left(1-a^2x^2\right)^{1/2}}{1/2}+c = x\cdot\sin^{-1}ax+\frac{\sqrt{1-a^2x^2}}{a}+c$$

8) let $u=e^{ax} \Rightarrow du=a\cdot e^{ax}\,dx$ & $dv=\sin bx\,dx \Rightarrow v=-\dfrac{1}{b}\cos bx$

$$\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b}\int e^{ax}\cdot\cos bx\,dx \quad ....(1)$$

let $u=e^{ax} \Rightarrow du=a\cdot e^{ax}\,dx$ & $dv=\cos bx\,dx \Rightarrow v=\dfrac{1}{b}\sin bx$

$$\int e^{ax}\cdot\cos bx\,dx=\frac{1}{b}e^{ax}\cdot\sin bx-\frac{a}{b}\int e^{ax}\cdot\sin bx\,dx \quad ....(2)$$

sub. (2) in (1) $\Rightarrow$

$$\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx\,dx-\frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx$$

$$\int e^{ax}\cdot\sin bx\,dx+\frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx\,dx+c$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cdot\sin bx\,dx=\frac{e^{ax}}{b^2}\left(a\sin bx-b\cos bx\right)+c$$

$$\therefore \quad \int e^{ax}\cdot\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)+c$$

9) <u>derivative of u</u> <u>integration of dv</u>

derivative of u	sign	integration of dv
x^3	+	e^x
$3x^2$	-	e^x
$6x$	+	e^x
6	-	e^x
0		e^x

$$\therefore \int x^3e^{ax}\,dx = x^3e^x-3x^2e^x+6xe^x-6e^x+c$$

$$=e^x\left(x^3-3x^2+6x-6\right)+c$$

$$10)\quad let\quad u = x^2 \Rightarrow du = 2x\,dx \quad \& \quad dv = x\cdot e^{x^2}dx \Rightarrow v = \frac{1}{2}e^{x^2}$$

$$\int x^3\cdot e^{x^2}dx = \frac{1}{2}x^2\cdot e^{x^2} - \frac{1}{2}\int 2x\cdot e^{x^2}dx = \frac{1}{2}x^2\cdot e^{x^2} - \frac{1}{2}e^{x^2} + c$$

6-2- Odd and even powers of sine and cosine:

To integrate an odd positive power of *sinx* (say $sin^{2n+1}x$) we split off a factor of *sinx* and rewrite the remaining even power in terms of the cosine. We write:-

$$\int sin^{2n+1}x\cdot dx = \int\left(1-cos^2 x\right)^n\cdot sin\,x\;dx$$

$$and\;\int cos^{2n+1}x\cdot dx = \int\left(1-sin^2 x\right)^n\cdot cos\,x\;dx$$

***EX-2*- Evaluate:**

$1)\ \int sin^3 x\,dx$ $\qquad\qquad 2)\ \int cos^5 x\,dx$

***Sol.*-**

$$1)\ \int sin^3x\,dx = \int sin^2x\cdot sinx\,dx = \int\left(1-cos^2x\right)\cdot sinx\,dx$$

$$= \int sinx\,dx + \int cos^2x\cdot\left(-sinx\right)dx = -cos\,x + \frac{1}{3}cos^3 x + c$$

$$2)\ \int cos^5x\,dx = \int cos^4x\cdot cosx\;dx = \int\left(1-sin^2x\right)^2\cdot cos\,x\,dx$$

$$= \int cosx\,dx - 2\int sin^2 x\cdot cos\,x\,dx + \int sin^4 x\cdot cos\,x\,dx$$

$$= sinx - \frac{2}{3}sin^3 x + \frac{1}{5}sin^5 x + c$$

To integrate an even positive power of sine (say $sin^{2n}x$) we use the relations:-

$$cos^2\theta = \frac{1+cos\,2\theta}{2} \qquad or \qquad sin^2\theta = \frac{1-cos\,2\theta}{2}$$

then we can write:-

$$\int \sin^{2n} x \cdot dx = \int \left(\frac{1-\cos 2x}{2} \right)^n \, dx$$

$$\text{and} \int \cos^{2n} x \cdot dx = \int \left(\frac{1+\cos 2x}{2} \right)^n \, dx$$

EX-3- **Evaluate:**

1) $\int \cos^2 \theta \, d\theta$ 2) $\int \sin^4 \theta \, d\theta$

Sol.-

$$1) \int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\left[\int d\theta + \frac{1}{2}\int 2\cos 2\theta \, d\theta \right]$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta \right] + c$$

$$2) \int \sin^4\theta \, d\theta = \int \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta = \frac{1}{4}\left[\int d\theta - \int \cos 2\theta (2d\theta) + \int \cos^2 2\theta \, d\theta \right]$$

$$= \frac{1}{4}\left[\theta - \sin 2\theta + \int \frac{1+\cos 4\theta}{2} d\theta \right] = \frac{1}{4}\left[\theta - \sin 2\theta + \frac{1}{2}(\theta + \frac{1}{4}\sin 4\theta) \right] + c$$

$$= \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c$$

To integrate the following identities:-

$$\int \sin mx \cdot \sin nx \, dx \quad , \int \sin mx \cdot \cos nx \, dx \quad ,\text{and} \int \cos mx \cdot \cos nx \, dx$$

we use the following formulas:-

$$\sin mx \cdot \sin nx = \frac{\cos(m-n)x - \cos(m+n)x}{2}$$

$$\sin mx \cdot \cos nx = \frac{\sin(m-n)x + \sin(m+n)x}{2}$$

$$\cos mx \cdot \cos nx = \frac{\cos(m-n)x + \cos(m+n)x}{2}$$

EX-4- **Evaluate:**

$$1)\ \int \sin 3x \cdot \cos 5x\, dx \qquad 2)\ \int \cos x \cdot \cos 7x\, dx \qquad 3)\int \sin x \cdot \sin 2x\, dx$$

Sol.-

$$1)\ \int \sin 3x \cdot \cos 5x\, dx = \frac{1}{2}\int \big(\sin(3x-5x) + \sin(3x+5x)\big)dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\int \sin 2x(2dx) + \frac{1}{8}\int \sin 8x(8dx)\right] = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c$$

$$2)\ \int \cos x \cdot \cos 7x\, dx = \frac{1}{2}\int \big(\cos(6x) + \cos(8x)\big)dx = \frac{1}{12}\sin 6x + \frac{1}{16}\sin 8x + c$$

$$3)\ \int \sin x \cdot \sin 2x\, dx = \frac{1}{2}\int \big(\cos x - \cos 3x\big)dx = \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$$

6-3- <u>Trigonometric substitutions</u>:

Trigonometric substitutions enable us to replace the binomials $a^2 - u^2$, $a^2 + u^2$, *and* $u^2 - a^2$ **be single square terms. We can use:-**

$$u = a\sin\theta \quad for \quad a^2 - u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

$$u = a\tan\theta \quad for \quad a^2 + u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$$u = a\sec\theta \quad for \quad u^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

EX-5 **Evaluate the following integrals:**

$$1)\ \int \frac{z^5\, dz}{\sqrt{1+z^2}} \qquad 4)\ \int \frac{x^2}{\sqrt{9-x^2}}dx$$

$$2)\ \int \frac{dx}{\sqrt{4+x^2}} \qquad 5)\ \int \frac{dt}{\sqrt{25t^2-9}}$$

$$3)\ \int \frac{dx}{4-x^2} \qquad 6)\ \int \frac{dy}{\sqrt{25+9y^2}}$$

Sol.-

1) let $z=\tan\theta \quad\Rightarrow\quad dz=\sec^2\theta\cdot d\theta \qquad\qquad \tan\theta=\dfrac{z}{1}$

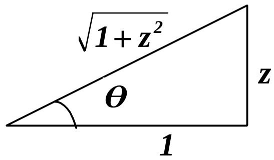
$$\int\frac{z^5\,dz}{\sqrt{1+z^2}}=\int\frac{\tan^5\theta\cdot\sec^2\theta\;d\theta}{\sqrt{1+\tan^2\theta}}=\int\tan^5\theta\cdot\sec\theta\;d\theta$$

$$=\int\tan\theta\cdot\sec\theta\left(\sec^2\theta-1\right)^2 d\theta$$

$$=\int\sec^4\theta\left(\tan\theta\cdot\sec\theta\,d\theta\right)-2\int\sec^2\theta\left(\tan\theta\cdot\sec\theta\,d\theta\right)+\int\tan\theta\cdot\sec\theta\,d\theta$$

$$=\frac{1}{5}\sec^5\theta-\frac{2}{3}\sec^3\theta+\sec\theta+c$$

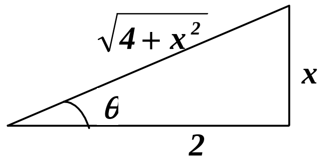
$$=\frac{1}{5}(\sqrt{1+z^2}\,)^5-\frac{2}{3}(\sqrt{1+z^2}\,)^3+\sqrt{1+z^2}+c$$



2) let $x=2\tan\theta \quad\Rightarrow\quad dx=2\sec^2\theta\cdot d\theta \qquad\qquad \tan\theta=\dfrac{x}{2}$

$$\int\frac{dx}{\sqrt{4+x^2}}=\int\frac{2\sec^2\theta\;d\theta}{\sqrt{4+4\tan^2\theta}}=\int\sec\theta\;d\theta=\ln\left|\sec\theta+\tan\theta\right|+c$$

$$=\ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right|+c$$

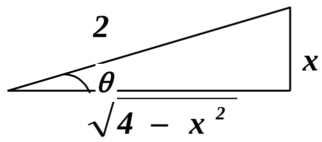


$$=\ln\left|\sqrt{4+x^2}+x\right|+c' \qquad \text{where } c'=c-\ln 2$$

3) let $x=2\sin\theta \quad\Rightarrow\quad dx=2\cos\theta\cdot d\theta$

$$\int\frac{dx}{4-x^2}=\int\frac{2\cos\theta\;d\theta}{4-4\sin^2\theta}=\frac{1}{2}\int\frac{d\theta}{\cos\theta}=\frac{1}{2}\int\sec\theta\,d\theta$$

$$=\frac{1}{2}\ln\left|\sec\theta+\tan\theta\right|+c$$



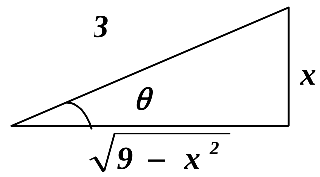
$$=\frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}}+\frac{x}{\sqrt{4-x^2}}\right|+c$$

$$=\frac{1}{2}\ln\left|\frac{2+x}{\sqrt{(2-x)(2+x)}}\right|+c=\frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right|+c=\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|+c$$

4) let $x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta \cdot d\theta$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$



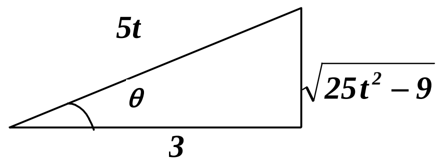
$$= \frac{9}{2}\left(\theta - \sin\theta\cdot\cos\theta\right) + c$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + c = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\cdot\sqrt{9-x^2} + c$$

5) let $5t = 3\sec\theta \quad \Rightarrow \quad 5dt = 3\sec\theta\cdot\tan\theta\,d\theta$

$$\int \frac{dt}{\sqrt{25t^2-9}} = \int \frac{\frac{3}{5}\sec\theta\cdot\tan\theta\,d\theta}{\sqrt{9\sec^2\theta-9}} = \frac{1}{5}\int \sec\theta\,d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + c$$



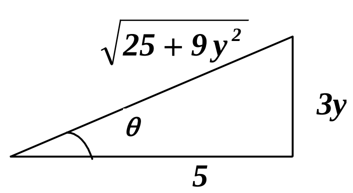
$$= \frac{1}{5}\ln\left|\frac{5t}{3} + \frac{\sqrt{25t^2-9}}{3}\right| + c$$

$$= \frac{1}{5}\ln\left|5t + \sqrt{25t^2-9}\right| + c' \quad \text{where } c' = c - \frac{1}{5}\ln 3$$

6) let $3y = 5\tan\theta \quad \Rightarrow \quad 3dy = 5\sec^2\theta\,d\theta$

$$\int \frac{dy}{\sqrt{25+9y^2}} = \int \frac{\frac{5}{3}\sec^2\theta\,d\theta}{\sqrt{25+25\tan^2\theta}} = \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + c$$



$$= \frac{1}{3}\ln\left|\frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5}\right| + c$$

$$= \frac{1}{3}\ln\left|\sqrt{25+9y^2} + 3y\right| + c' \quad \text{where } c' = c - \frac{1}{3}\ln 5$$

EX-6 **Prove the following formulas:**

$$1)\ \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c \qquad 2)\ \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

Proof.-

$$1)\quad let \quad u = a\sin\theta \quad \Rightarrow \quad du = a\cos\theta \cdot d\theta$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{a\cos\theta \cdot d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}\frac{u}{a} + c$$

$$2)\quad let \quad u = a\tan\theta \quad \Rightarrow \quad du = a\sec^2\theta \cdot d\theta$$

$$\int \frac{du}{a^2+u^2} = \int \frac{a\sec^2\theta \cdot d\theta}{a^2+a^2\tan^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

6-4- Integral involving $a x^2 + b x + c$:

By using the algebraic process called completing the square, we can convert any quadratic: $a x^2 + b x + c$, $a \neq 0$ to the form: $a(u^2 \mp A^2)$ we can then use one of the trigonometric substitutions to write the expression as a times a single square term.

EX-7 **– Evaluate:**

$$1)\ \int \frac{dx}{\sqrt{2x-x^2}} \qquad 4)\ \int \frac{dx}{\sqrt{1+x-x^2}}$$

$$2)\ \int \frac{dx}{2x^2+2x+1} \qquad 5)\ \int \frac{dx}{\sqrt{x^2-2x-8}}$$

$$3)\ \int \frac{dx}{\sqrt{x^2+2x+2}}$$

Sol.

$$1)\ \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

$$let \quad x-1 = \sin\theta \quad \Rightarrow \quad dx = \cos\theta\, d\theta$$

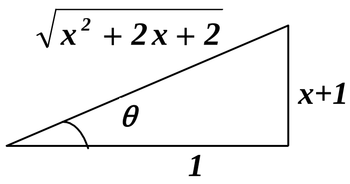
$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}(x-1) + c$$

$$2)\ \int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{dx}{x^2+x+\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$let \quad x+\frac{1}{2} = \frac{1}{2}\tan\theta \quad \Rightarrow \quad dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$\int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\frac{1}{4}\tan^2\theta+\frac{1}{4}} = \int d\theta = \theta + c = \tan^{-1}(2x+1)+c$$

$$3)\ \int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{dx}{\sqrt{(x+1)^2+1}}$$



$$let \quad x+1 = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\, d\theta$$

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{\tan^2\theta+1}} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta+\tan\theta\right| + c = \ln\left|\sqrt{x^2+2x+2}+x+1\right| + c$$

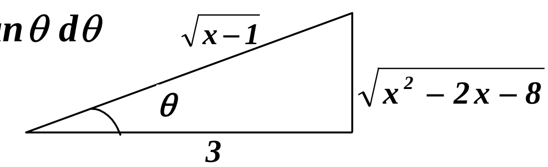
$$4)\ \int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{dx}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}}$$

$$let \quad x-\frac{1}{2} = \frac{\sqrt{5}}{2}\sin\theta \quad \Rightarrow \quad dx = \frac{\sqrt{5}}{2}\cos\theta\, d\theta$$

$$= \int \frac{\frac{\sqrt{5}}{2}\ \cos\theta\, d\theta}{\sqrt{\frac{5}{4}-\frac{5}{4}\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$$

$$5)\ \int \frac{dx}{\sqrt{x^2-2x-8}} = \int \frac{dx}{\sqrt{(x-1)^2-9}}$$

$$let \ \ x-1 = 3\sec\theta \quad \Rightarrow \quad dx = 3\sec\theta\cdot\tan\theta\, d\theta$$



$$= \int \frac{3\sec\theta\cdot\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta+\tan\theta\right| + c = \ln\left|\frac{x-1}{3}+\frac{\sqrt{x^2-2x-8}}{3}\right| + c$$

$$= \ln\left|x-1+\sqrt{x^2-2x-8}\right| + c' \qquad where \quad c' = c - \ln 3$$

6-5- Partial fractions:

Success in separating $\frac{f(x)}{g(x)}$ into a sum of partial fractions hinges on two things:-

1- **The degree of $f(x)$ must be less than the degree of $g(x)$. (If this is not case, we first perform a long division, and then work with the remainder term).**

2- **The factors of $g(x)$ must be known. If these two conditions are met we can carry out the following steps:**

Step I - let $x-r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $(x-r)$ that divides $g(x)$. Then assign the sum of m partial factors to this factor, as follows:

$$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\dots\dots\dots+\frac{A_m}{(x-r)^m}$$

Do this for each distinct linear factor of $f(x)$.

Step II - let x^2+px+q be an irreducible quadratic factor of $g(x)$. Suppose $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1\,x+C_1}{x^2+px+q}+\frac{B_2\,x+C_2}{(x^2+px+q)^2}+\dots\dots\dots+\frac{B_n\,x+C_n}{(x^2+px+q)^n}$$

Do this for each distinct linear factor of $g(x)$.

Step III - set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the sums in decreasing powers of x.

Step IV - equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EX-8 **– Evaluate the following integrals:**

1) $\int \frac{2x+5}{x^2-9}\,dx$

2) $\int \frac{x\,dx}{x^2+4x+3}$

3) $\int \frac{x^3-x}{(x^2+1)\cdot(x-1)^2}\,dx$

4) $\int \frac{\sin x\;dx}{\cos^2 x-5\cos x+4}$

5) $\int \frac{2x^2-3x+2}{(x-1)^2(x-2)}\,dx$

6) $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx$

Sol.-

1) $\int \frac{2x+5}{x^2-9}\,dx = \int \frac{2x+5}{(x-3)\cdot(x+3)}\,dx$

$$\frac{2x+5}{(x-3)\cdot(x+3)} = \frac{A}{x-3}+\frac{B}{x+3} \quad\Rightarrow\quad 2x+5 = A(x+3)+B(x-3)$$

$$at \quad x=3 \quad\Rightarrow\quad 6A = 6+5 \quad\Rightarrow\quad A=\frac{11}{6}$$

$$at \quad x=-3 \quad\Rightarrow\quad -6B = -6+5 \quad\Rightarrow\quad B=\frac{1}{6}$$

$$\int \frac{2x+5}{x^2-9}\,dx = \int\left(\frac{11/6}{x-3}+\frac{1/6}{x+3}\right)dx = \frac{11}{6}\ln(x-3)+\frac{1}{6}\ln(x+3)+c$$

2) $\int \frac{x\,dx}{x^2+4x+3} = \int \frac{x\,dx}{(x+3)(x+1)}$

$$\frac{x}{(x+3)(x+1)} = \frac{A}{x+3}+\frac{B}{x+1} \quad\Rightarrow\quad x = A(x+1)+B(x+3)$$

$$at \quad x=-3 \quad\Rightarrow\quad A=\frac{3}{2} \quad and \quad at \quad x=-1 \quad\Rightarrow\quad B=-\frac{1}{2}$$

$$\int \frac{x\,dx}{x^2+4x+3} = \int\left(\frac{3/2}{x+3}+\frac{-1/2}{x+1}\right)dx = \frac{3}{2}\ln(x+3)-\frac{1}{2}\ln(x+1)+c$$

3) $\int \frac{x^3 - x}{(x^2+1)(x-1)^2}dx = \int \frac{x(x-1)(x+1)}{(x^2+1)(x-1)^2}dx = \int \frac{x^2+x}{(x^2+1)(x-1)}dx$

$$\frac{x^2+x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} \quad \Rightarrow \quad x^2+x = (Ax+B)(x-1) + C(x^2+1)$$

$$x^2 + x = (A+C)x^2 + (-A+B)x + (-B+C)$$

$$\left.\begin{array}{l} A+C=1 \quad ......(1) \\ -A+B=1 \quad ......(2) \\ -B+C=0 \quad ......(3) \end{array}\right\} \Rightarrow \quad A=0 \quad , \quad B=1 \; , \; C=1$$

$$\int \frac{x^3-x}{(x^2+1)(x-1)^2}dx = \int \left(\frac{1}{x^2+1} + \frac{1}{x-1} \right) dx = tan^{-1} x + ln(x-1) + c$$

4) let $y = cosx \quad \Rightarrow \quad dy = -sinx\, dx$

$$\int \frac{sin\,x\;dx}{cos^2 x - 5\cos x + 4} = -\int \frac{dy}{y^2 - 5y + 4} = -\int \frac{dy}{(y-4)(y-1)}$$

$$\frac{dy}{(y-4)(y-1)} = \frac{A}{y-4} + \frac{B}{y-1} \quad \Rightarrow \quad 1 = A(y-1) + B(y-4)$$

at $y = 4 \quad \Rightarrow \quad A = \frac{1}{3}$ and at $y = 1 \quad \Rightarrow \quad B = -\frac{1}{3}$

$$\int \frac{sin\,x\;dx}{cos^2 x - 5\cos x + 4} = -\int \left(\frac{1/3}{y-4} + \frac{-1/3}{y-1} \right) dy$$

$$= -\frac{1}{3} ln(y-4) + \frac{1}{3} ln(y-1) + c = -\frac{1}{3} ln(cos\,x - 4) + \frac{1}{3} ln(cos\,x - 1) + c$$

5) $\frac{2x^2 - 3x + 2}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$$2x^2 - 3x + 2 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$$\left.\begin{array}{l} A+C=2 \quad ....(1) \\ -3A+B-2C=-3 \;......(2) \\ 2A-2B+C=2 \;...........(3) \end{array}\right\} \Rightarrow \quad A=-2 \quad , \quad B=-1 \; , \; C=4$$

$$\int \frac{2x^2-3x+2}{(x-1)^2(x-2)}dx = \int \left(\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{4}{x-2} \right) dx$$

$$= -2\,ln(x-1) + \frac{1}{x-1} + 4\,ln(x-2) + C$$

6) $\dfrac{x^3+4x^2}{x^2+4x+3} = x - \dfrac{3x}{(x+3)(x+1)}$

$$\begin{array}{r|l} & x \\ \hline x^2+4x+3 & x^3+4x^2 \\ & \mp x^3 \mp 4x^2 \mp 3x \\ \hline & -3x \end{array}$$

$$\frac{3x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} \quad \Rightarrow \quad 3x = A(x+1) + B(x+3)$$

$$at \;\; x=-3 \Rightarrow A = \frac{9}{2} \quad and \quad at \;\; x=-1 \Rightarrow B = -\frac{3}{2}$$

$$\int \frac{x^3+4x^2}{x^2+4x+3}\,dx = \int \left(x - \frac{9/2}{x+3} + \frac{3/2}{x+1} \right) dx$$

$$= \frac{x^2}{2} - \frac{9}{2}\ln(x+3) - \frac{3}{2}\ln(x+1) + c$$

6-6- Rational functions of *sinx* and *cosx*, and other trigonometric integrals:

We assume that $z = \tan\frac{x}{2}$ then $x = 2\tan^{-1} z$ and $dx = \frac{2}{1+z^2}dz$

Since

$$\cos^2\frac{x}{2} = \frac{1+\cos x}{2} \quad \Rightarrow \quad \cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\frac{x}{2}} - 1$$

$$= \frac{2}{\tan^2\frac{x}{2}+1} - 1 = \frac{2}{z^2+1} - 1 \quad \Rightarrow \quad \cos x = \frac{1-z^2}{1+z^2}$$

Since

$$\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}\cdot\cos^2\frac{x}{2} = 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2\frac{x}{2}}$$

$$= 2\tan\frac{x}{2}\cdot\frac{1}{\tan^2\frac{x}{2}+1} \quad \Rightarrow \quad \sin x = \frac{2z}{1+z^2}$$

EX-9 **– Evaluate:**

1) $\int \frac{dx}{1+\sin x+\cos x}$

2) $\int \frac{dx}{\sin x+\tan x}$

3) $\int \frac{dx}{2+\sin x}$

4) $\int \frac{3\ dx}{2+4\sin x}$

5) $\int secx\,dx$

6) $\int \frac{\cos x\,dx}{1-\cos x}$

Sol.-

1) $$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2}{1+z^2}dz}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}} = \int \frac{dz}{1+z}$$

$$= \ln|1+z|+c = \ln\left|1+\tan\frac{x}{2}\right|+c$$

2) $$\int \frac{dx}{\sin x+\tan x} = \int \frac{\frac{2}{1+z^2}dz}{\frac{2z}{1+z^2}+\frac{2z}{1-z^2}} = \frac{1}{2}\int\left(\frac{1}{z}-z\right)dz$$

$$= \frac{1}{2}\left[\ln z-\frac{z^2}{2}\right]+c = \frac{1}{2}\left[\ln\tan\frac{x}{2}-\frac{1}{2}\tan^2\frac{x}{2}\right]+c$$

3) $$\int \frac{dx}{2+\sin x} = \int \frac{\frac{2}{1+z^2}dz}{2+\frac{2z}{1+z^2}} = \int \frac{dz}{z^2+z+1} = \int \frac{dz}{(z+\frac{1}{2})^2+\frac{3}{4}}$$

$$let \quad z+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta \ \Rightarrow\ dz = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$$

$$\int \frac{dx}{2+\sin x} = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{3}{4}\tan^2\theta+\frac{3}{4}} = \frac{2}{\sqrt{3}}\int d\theta = \frac{2}{\sqrt{3}}\theta+c$$

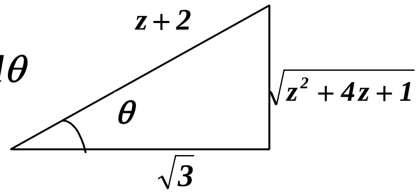
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right)+c$$

$$4)\ \int \frac{3\ dx}{2+4\sin x} = \frac{3}{2}\int \frac{dx}{1+2\sin x} = \frac{3}{2}\int \frac{\frac{2}{z^2+1}dz}{1+2\frac{2z}{z^2+1}} = 3\int \frac{dz}{z^2+4z+1}$$

$$= 3\int \frac{dz}{(z+2)^2-3} = \int \frac{dz}{(\frac{z+2}{\sqrt{3}})^2-1}$$

$$let \quad \frac{z+2}{\sqrt{3}} = \sec\theta \quad \Rightarrow \quad dz = \sqrt{3}\sec\theta\cdot\tan\theta\, d\theta$$

$$\int \frac{3\ dx}{2+4\sin x} = \int \frac{\sqrt{3}\sec\theta\cdot\tan\theta\, d\theta}{\sec^2\theta-1} = \sqrt{3}\int \frac{\sec\theta}{\tan\theta}d\theta$$



$$= \sqrt{3}\int \csc\theta\, d\theta = -\sqrt{3}\ln\left|\csc\theta+\cot\theta\right| + c$$

$$= -\sqrt{3}\ln\left|\frac{z+2}{\sqrt{z^2+4z+1}} + \frac{\sqrt{3}}{\sqrt{z^2+4z+1}}\right| + c = -\sqrt{3}\ln\left|\frac{\tan\frac{x}{2}+2+\sqrt{3}}{\sqrt{\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1}}\right| + c$$

$$5)\ \int secx\, dx = \int \frac{1+z^2}{1-z^2}\cdot\frac{2}{1+z^2}dz = 2\int \frac{1}{(1-z)(1+z)}dz$$

$$\frac{1}{(1-z)(1+z)} = \frac{A}{1-z} + \frac{B}{1+z} \quad \Rightarrow \quad A(1+z) + B(1-z) = 1$$

$$at \quad z=1 \quad \Rightarrow \quad A = \frac{1}{2} \quad and \quad at \quad z=-1 \quad \Rightarrow \quad B = \frac{1}{2}$$

$$\int secx\, dx = 2\int \left(\frac{1/2}{1-z} + \frac{1/2}{1+z}\right)dz = -\ln(1-z) + \ln(1+z) + c$$

$$= \ln\left(1+\tan\frac{x}{2}\right) - \ln\left(1-\tan\frac{x}{2}\right) + c = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + c$$

$$By\ substituting \quad \tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} \quad implies$$

$$\int secx\, dx = \ln\left|\sec x + \tan x\right| + c$$

$$6)\ \int \frac{\cos x\,dx}{1-\cos x} = \int \frac{\frac{1-z^2}{1+z^2}}{1-\frac{1-z^2}{1+z^2}} \cdot \frac{2}{1+z^2}\,dz = \int \frac{1-z^2}{(1+z^2)\,z^2}\,dz$$

$$\frac{1-z^2}{(1+z^2)\,z^2} = \frac{A}{z} + \frac{B}{z^2} + \frac{C\,z+D}{1+z^2} \qquad \Rightarrow$$

$$Az + Az^3 + B + Bz^2 + Cz^3 + Dz^2 = 1 - z^2$$

$$\left.\begin{array}{l} A+C=0\ \ ......(1) \\ B+D=-1\ \ ....(2) \\ A=0\ \ ....(3) \\ B=1\ \ ....(4) \end{array}\right\} \Rightarrow \quad A=0\ \ ,\ \ B=1\ ,\ C=0\ ,\ D=-2$$

$$\int \frac{\cos x\,dx}{1-\cos x} = \int \left(\frac{1}{z^2} - \frac{2}{z^2+1} \right) dz = -\frac{1}{z} - 2\tan^{-1} z + c$$

$$= -\frac{1}{\tan\frac{x}{2}} - 2 \cdot \frac{x}{2} + c = -\cot\frac{x}{2} - x + c$$

Problems – 6

Evaluate the following integrals:

1) $\int \frac{x^3}{x-1}\,dx$ (ans.: $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + ln(x-1) + c$)

2) $\int \frac{3x+2}{3x-1}\,dx$ (ans.: $x + ln(3x-1) + c$)

3) $\int x^2 \cdot e^{-x}\,dx$ (ans.: $-e^{-x}(x^2+2x+2)+c$)

4) $\int x \cdot sin\,x^2\,dx$ (ans.: $-\frac{1}{2}cos\,x^2 + c$)

5) $\int \sqrt{x^2-1}\,dx$ (ans.: $\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}ln\left|x+\sqrt{x^2+1}\right| + c$)

6) $\int \frac{3x+13}{(5x-1)(7x+2)}$ (ans.: $\frac{4}{5}ln|5x-1| - \frac{5}{7}ln|7x+2| + c$)

7) $\int \frac{2x-3}{(x-1)(x-2)(x+3)}\,dx$ (ans.: $\frac{1}{4}ln|x-1| + \frac{1}{5}ln|x-2| - \frac{9}{20}ln|x+3| + c$)

8) $\int \frac{dx}{x^4-1}$ (ans.: $\frac{1}{4}ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}tan^{-1}\,x + c$)

9) $\int ln\,x\,dx$ (ans.: $x \cdot lnx - x + c$)

10) $\int tan^{-1}\,x\,dx$ (ans.: $x \cdot tan^{-1}x - \frac{1}{2}ln(1+x^2) + c$)

11) $\int x \cdot ln\,x\,dx$ (ans.: $\frac{x^2}{2}ln\,x - \frac{x^2}{4} + c$)

12) $\int x \cdot tan^{-1}x\,dx$ (ans.: $\frac{x^2}{2}tan^{-1}\,x - \frac{1}{2}(x - tan^{-1}\,x) + c$)

13) $\int x^2 \cdot cosax\,dx$ (ans.: $\frac{x^2}{a}sin\,ax + \frac{2x}{a^2}cos\,ax - \frac{2}{a^3}sin\,ax + c$)

14) $\int sin(lnx)\,dx$ (ans.: $\frac{x}{2}(sin(ln\,x) - cos(ln\,x)) + c$)

15) $\int ln(a^2+x^2)dx$ (ans.: $x \cdot ln(a^2+x^2) - 2x + 2a\,tan^{-1}\frac{x}{a} + c$)

16) $\int x \cdot \sin^{-1} x \, dx$ (ans.: $\frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c$)

17) $\int \cos^4 x \, dx$ (ans.: $\frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$)

18) $\int \cos^{\frac{2}{3}} x \cdot \sin^5 x \, dx$ (ans.: $-\frac{3}{5}\cos^{\frac{5}{3}} x + \frac{6}{11}\cos^{\frac{11}{3}} x - \frac{3}{17}\cos^{\frac{17}{3}} x + c$)

19) $\int x \cdot \sin x \, dx$ (ans.: $-x \cdot \cos x + \sin x + c$)

20) $\int x^2 \sqrt{1-x} \, dx$ (ans.: $-\frac{2}{105}\sqrt{(1-x)^3}\left(15x^2 + 12x + 8\right) + c$)

21) $\int \sin^2 x \cdot \cos^2 x \, dx$ (ans.: $\frac{1}{32}\left(4x - \sin 4x\right) + c$)

22) $\int \sec^3 x \cdot \tan^2 x \, dx$ (ans.: $\frac{1}{4}\sec^3 x \cdot \tan x - \frac{1}{8}\sec x \cdot \tan x$

$- \frac{1}{8}\ln\left|\sec x + \tan x\right| + c$)

23) $\int x\left(\cos^3 x^2 - \sin^3 x^2\right) dx$ (ans.: $\frac{1}{2}\sin x^2 - \frac{1}{6}\sin^3 x^2 + \frac{1}{2}\cos x^2$

$- \frac{1}{6}\cos^3 x^2 + c$)

24) $\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$ (ans.: $2\sin^{-1}\sqrt{x} + c$)

25) $\int \frac{dx}{\sqrt{x} \cdot (1+\sqrt{x})}$ (ans.: $2\ln(1+\sqrt{x}) + c$)

26) $\int \frac{dx}{x\sqrt{2 - 3\ln^2 \sqrt{x}}}$ (ans.: $\frac{2}{\sqrt{3}}\sin^{-1}(\frac{\sqrt{3}}{2}\ln x) + c$)

27) $\int \frac{e^{2x} \, dx}{\sqrt[3]{1+e^x}}$ (ans.: $\frac{3}{2} \cdot e^x \cdot \sqrt[3]{(1+e^x)^2} - \frac{9}{10}\sqrt[3]{(1+e^x)^5} + c$)

28) $\int \frac{dy}{y(2y^3+1)^2}$ (ans.: $\frac{1}{3}\ln(\frac{2y^3}{2y^3+1}) - \frac{2y^3}{3(2y^3+1)} + c$)

29) $\int \frac{x \, dx}{1+\sqrt{x}}$ (ans.: $\frac{2}{3}\sqrt{x^3} - x + 2\sqrt{x} - 2\ln(\sqrt{x}+1) + c$)

30) $\int \frac{dt}{e^t - 1}$ (ans.: $\ln(e^t - 1) - t + c$)

31) $\int \frac{d\theta}{1-\tan^2\theta}$ (ans.: $\frac{1}{2}\theta + \frac{1}{4}\ln\left|\sec 2\theta + \tan 2\theta\right| + c$)

32) $\int e^x \cdot \cos 2x \; dx$ (ans.: $\frac{e^x}{5}\cos 2x + \frac{2}{5}e^x \sin 2x + c$)

33) $\int \frac{\cot\theta \, d\theta}{1+\sin^2\theta}$ (ans.: $\ln\frac{\sin\theta}{\sqrt{1+\sin^2\theta}} + c$)

34) $\int \frac{e^{4t}}{(1+e^{2t})^{\frac{2}{3}}} dt$ (ans.: $\frac{3}{2}e^{2t}(1+e^{2t})^{\frac{1}{3}} - \frac{9}{8}(1+e^{2t})^{\frac{3}{4}} + c$)

35) $\int \frac{x^3+x^2}{x^2+x-2} dx$ (ans.: $\frac{x^2}{2} + \frac{4}{3}\ln(x+2) + \frac{2}{3}\ln(x-1) + c$)

36) $\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}} \; dx$ (ans.: $\frac{1}{3}(2\sqrt{3e^{2x}-6e^x-1} +$

$\sqrt{3}\ln\left|\sqrt{3}(e^x-1) + \sqrt{3e^{2x}-6e^x-1}\right| + c$)

37) $\int \frac{dy}{(2y+1)\sqrt{y^2+y}}$ (ans.: $\sec^{-1}(2y+1) + c$)

38) $\int (1-x^2)^{\frac{3}{2}} \; dx$ (ans.: $\frac{e^x}{5}\cos 2x + \frac{2}{5}e^x \sin 2x + c$)

39) $\int \frac{\tan^{-1} x}{x^2} dx$ (ans.: $\ln\frac{x}{\sqrt{x^2+1}} - \frac{\tan^{-1} x}{x^2} + c$)

40) $\int x \cdot \sin^2 x \; dx$ (ans.: $\frac{x^2}{4} - \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + c$)

41) $\int \frac{dt}{t^4+4t^2+3}$ (ans.: $\frac{1}{2}\tan^{-1} t - \frac{1}{2\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + c$)

42) $\int \frac{8dx}{x^4+2x^3}$ (ans.: $\ln\frac{x}{x+2} + \frac{2}{x} - \frac{2}{x^2} + c$)

43) $\int \frac{\cos x \; dx}{\sqrt{1+\cos x}}$ (ans.: $\sqrt{2}(2\sin\frac{x}{2} - \ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right|) + c$)

44) $\int \frac{x \; dx}{x+\sqrt{x}+1}$ (ans.: $x - 2\sqrt{x} + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x}+1}{\sqrt{3}} + c$)

45) $\int \frac{dt}{\sec^2 t + \tan^2 t}$ (ans.: $\sqrt{2}\tan^{-1}(\sqrt{2}\tan t) - t + c$)

$46)\ \int \frac{dx}{1+\cos^2 x}$ $\qquad (ans.:\ \frac{1}{\sqrt{2}}\tan^{-1}(\frac{1}{\sqrt{2}}\tan x)+c)$

$47)\ \int \ln(\sqrt{x}+\sqrt{1+x})\,dx$ $\qquad (ans.:\ x\ln(\sqrt{x}+\sqrt{1+x})-\frac{\sqrt{x^2+x}}{2}$

$+\frac{1}{4}\ln\left|2x+1+2\sqrt{x^2+x}\right|+c)$

$48)\ \int x\ln(x^3+x)\,dx$ $\qquad (ans.:\ \frac{x^2}{2}\ln(x^3+x)-\frac{3}{4}x^2+\frac{1}{2}\ln(x^2+1)+c)$

$49)\ \int \frac{\cos x\,dx}{\sqrt{4-\cos^2 x}}$ $\qquad (ans.:\ \ln\left|\sqrt{3+\sin^2 x}+\sin x\right|+c)$

$50)\ \int \frac{\sec^2 x\,dx}{\sqrt{4-\sec^2 x}}$ $\qquad (ans.:\ \sin^{-1}(\frac{1}{\sqrt{3}}\tan x)+c)$

$51)\ \int \frac{dt}{t-\sqrt{1-t^2}}$ $\qquad (ans.:\ \frac{1}{2}\ln(t-\sqrt{1-t^2})-\frac{1}{2}\sin^{-1} t+c)$

$52)\ \int e^{-x}\cdot\tan^{-1} e^x\,dx$ $\qquad (ans.:\ -e^{-x}\cdot\tan^{-1} e^x+x-\frac{1}{2}\ln(1+e^{2x})+c)$

$53)\ \int \sin^{-1}\sqrt{x}\,dx$ $\qquad (ans.:\ x\sin^{-1}\sqrt{x}-\frac{1}{2}\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+c)$

$54)\ \int \frac{\cos 2x-1}{\cos 2x+1}\,dx$ $\qquad (ans.:\ x-\tan x+c)$

Chapter seven

Application of integrals

7-1- Definite integrals:

If $f(x)$ is continuous in the interval $a \le x \le b$ and it is integrable in the interval then the area under the curve:-

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:-

1) $\int_a^b c\ f(x)\,dx = c\int_a^b f(x)\,dx$, where c is constant.

2) $\int_a^b \left(f(x) \mp g(x) \right)dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) Let $a < c < b$ then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5) $\int_a^a f(x)\,dx = 0$

6) If $f(x) \ge 0$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \ge 0$

7) If $f(x) \le g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

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4) *Let* $a < c < b$ *then* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5) $\int_a^a f(x)\,dx = 0$

6) *If* $f(x) \ge 0$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \ge 0$

7) *If* $f(x) \le g(x)$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

EX-1 **– Evaluate the following definite integrals:**

1) $\int_{2}^{6}\frac{dx}{x+2}$

2) $\int_{\pi/2}^{3\pi/2}\cos x\,dx$

3) $\int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{1+x^2}$

4) $\int_{0}^{\sqrt{3}/2}\frac{dx}{\sqrt{1-x^2}}$

5) $\int_{-2}^{4}e^{-\frac{x}{2}}\,dx$

6) $\int_{0}^{\pi}(\pi-x)\cdot\cos x\,dx$

Sol. **–**

$$1)\ \int_{2}^{6}\frac{dx}{x+2}=\ln(x+2)\Big|_{2}^{6}=\ln(6+2)-\ln(2+2)=\ln 8-\ln 4=3\ln 2-2\ln 2=\ln 2$$

$$2)\ \int_{\pi/2}^{3\pi/2}\cos x\,dx=\sin x\Big|_{\pi/2}^{3\pi/2}=\sin(\frac{3}{2}\pi)-\sin(\frac{\pi}{2})=-1-1=-2$$

$$3)\ \int_{-\sqrt{3}}^{\sqrt{3}}\frac{dx}{1+x^2}=\tan^{-1}\Big|_{-\sqrt{3}}^{\sqrt{3}}=\tan^{-1}\sqrt{3}-\tan^{-1}(-\sqrt{3})=\frac{\pi}{3}-(-\frac{\pi}{3})=\frac{2}{3}\pi$$

$$4)\ \int_{0}^{\sqrt{3}/2}\frac{dx}{\sqrt{1-x^2}}=\sin^{-1}x\Big|_{0}^{\sqrt{3}/2}=\sin^{-1}\frac{\sqrt{3}}{2}-\sin^{-1}0=\frac{\pi}{3}-0=\frac{\pi}{3}$$

$$5)\ \int_{-2}^{4}e^{-\frac{x}{2}}\,dx=-2e^{-\frac{x}{2}}\Big|_{-2}^{4}=-2(e^{-2}-e)=2(e-e^{-2})$$

$$6)\ \text{Let}\quad u=\pi-x\ \Rightarrow\ du=-dx\quad\&\quad dv=\cos x\,dx\ \Rightarrow\ v=\sin x$$

$$\int_{0}^{\pi}(\pi-x)\cdot\cos x\,dx=(\pi-x)\sin x\Big|_{0}^{\pi}+\int_{0}^{\pi}\sin x\,dx=(\pi-x)\sin x-\cos x\Big|_{0}^{\pi}$$

$$=(\pi-\pi)\sin\pi-\cos\pi-\big((\pi-0)\sin 0-\cos 0\big)=0-(-1)-(0-1)=2$$

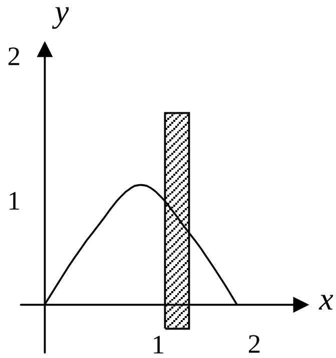
7-2- Area between two curves:

Suppose that $y_1 = f_1(x)$ and $y_2 = f_2(x)$ define two functions of x that are continuous for $a \le x \le b$ then the area bounded above by the y_1 curve, below by y_2 curve and on the sides by the vertical lines $x = a$ and $x = b$ is:-

$$A = \int_a^b \left[f_1(x) - f_2(x) \right] dx$$

***EX-2*- Find the area bounded by the *x*-axis and the curve:**

$$y = 2x - x^2$$



***Sol.*-**

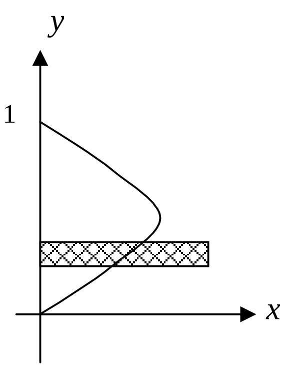
$$\left.\begin{aligned} y &= 0 \quad \ldots\ldots\ldots\ldots(1) \\ y &= 2x - x^2 \quad \ldots\ldots(2) \end{aligned}\right\} \Rightarrow \; x(x-2) = 0 \;\Rightarrow\; x = 0\,,2$$

The points of the intersection of the curve and the *x*-axis are (0,0) and (2,0) then the area bounded by *x*-axis and the curve is:-

$$\int_0^2 (2x - x^2)\,dx = x^2 - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} - (0-0) = \frac{4}{3}$$

***EX-3*- Find the area bounded by the *y*-axis and the curve:**

$$x = y^2 - y^3$$



***Sol.*-**

$$\left.\begin{aligned} x &= 0 \quad \ldots\ldots\ldots\ldots(1) \\ x &= y^2 - y^3 \quad \ldots\ldots(2) \end{aligned}\right\} \Rightarrow \; y^2(1-y) = 0 \;\Rightarrow\; y = 0\,,1$$

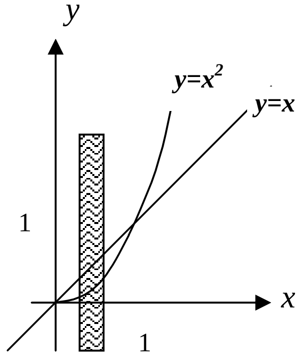
$$\Rightarrow \; intersection \; points \;\; (0,0),(0,1)$$

The area =

$$A = \int_0^1 (y^2 - y^3)\,dy = \frac{y^3}{3} - \frac{y^4}{4}\Big|_0^1 = \frac{1}{3} - \frac{1}{4} - (0-0) = \frac{1}{12}$$

EX-4- **Find the area bounded by the curve** $y = x^2$ **and the line:** $y = x$

Sol.-



$$\left.\begin{array}{l} y = x^2 \text{ ......(1)} \\ y = x \text{ .....(2)} \end{array}\right\} \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$$

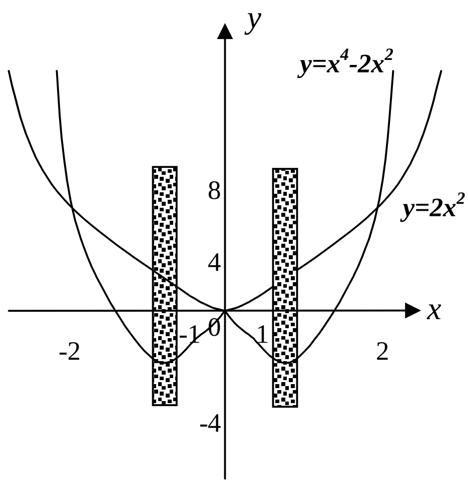
$$\Rightarrow \textit{intersection points } (0,0), (1,1)$$

The area =

$$A = \int_0^1 (x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}\Bigg|_0^1 = \frac{1}{2} - \frac{1}{3} - 0 = \frac{1}{6}$$

EX-5- **Find the area bounded by the curves** $y = x^4 - 2x^2$ **and** $y = 2x^2$

Sol.-



$$\left.\begin{array}{l} y = x^4 - 2x^2 \text{ ......(1)} \\ y = 2x^2 \text{ ....(2)} \end{array}\right\} \Rightarrow x^2(x^2 - 4) = 0$$

$$\Rightarrow x = 0, 2, -2$$

$$\Rightarrow \textit{intersection points are } (0,0), (2,8), (-2,8)$$

The area =

$$A = \int_{-2}^{0} \left(2x^2 - (x^4 - 2x^2)\right)dx + \int_0^2 \left(2x^2 - (x^4 - 2x^2)\right)dx$$

$$= 2\int_0^2 \left(4x^2 - x^4\right)dx = 2\left[\frac{4}{3}x^3 - \frac{x^5}{5}\right]_0^2 = 2\left[\frac{4}{3}\cdot 8 - \frac{32}{5} - 0\right]$$

$$= \frac{128}{15}$$

Notice:- We can use the double integration to calculate the area between two curves which bounded above by the curve $y = f_2(x)$ below by $y = f_1(x)$ on the left by the line $x = a$ and on the right by $x = b$, then:-

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

To evaluate above integrals we follow:-

(a) integrating $\int dy$ with respect to y and evaluating the resulting integral the limits $y = f_1(x)$ and $y = f_2(x)$, then:

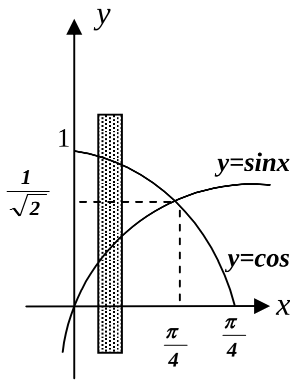
(b) integrating the result of (a) with respect to x between the limits $x = a$ and $x = b$.

If the area is bounded on the left by the curve $x = g_1(y)$, on the right by $x = g_2(y)$, below by the line $y = c$, and above by the line $y = d$, then it is better to integrate first with respect to x and then with respect to y. That is:-

$$A = \int_c^d \int_{g_1(y)}^{g_2(y)} dx\,dy$$

EX-6- Find the area of the triangular region in the first quadrant bounded by the y-axis and the curve $y = \sin x$, $y = \cos x$.

Sol.-



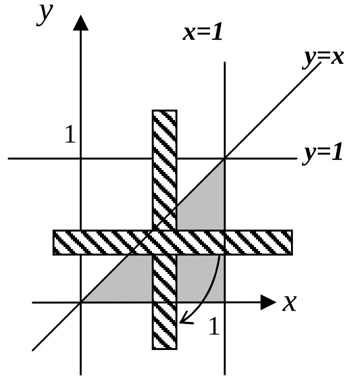
$$\left.\begin{array}{l} y = \sin x \;\; .....(1) \\ y = \cos x \;\; .....(2) \end{array}\right\} \Rightarrow \sin x = \cos x \qquad \therefore x = \frac{\pi}{4}$$

The area =

$$A = \int_0^{\frac{\pi}{4}} \int_{\sin x}^{\cos x} dy\,dx = \int_0^{\frac{\pi}{4}} y \Big|_{\sin x}^{\cos x} dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$$

$$= \sin x + \cos x \Big|_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0 + 1) = \sqrt{2} - 1 = 0.414$$

EX-7- Calculate: $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$



Sol.- We cannot solve the integration $\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy$, hence we reverse the order of integration as follow:-

$x = 1$ and $y = 1$

$x = y$ $y = 0$

$$A = \int_0^1 \int_0^x \frac{\sin x}{x} dy\, dx = \int_0^1 \frac{\sin x}{x} y\Big|_0^x\, dx = \int_0^1 \frac{\sin x}{x}(x - 0)dx$$

$$= \int_0^1 \sin x\, dx = -\cos x\Big|_0^1 = -(\cos 1 - \cos 0) = 1 - \cos 1$$

EX-8- Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.

$$1)\ \int_{-2}^{1} \int_{x^2+4x}^{3x+2} dy\, dx \qquad 2)\ \int_{-1}^{0} \int_{-2x}^{1-x} dy\, dx + \int_{0}^{2} \int_{-\frac{x}{2}}^{1-x} dy\, dx$$

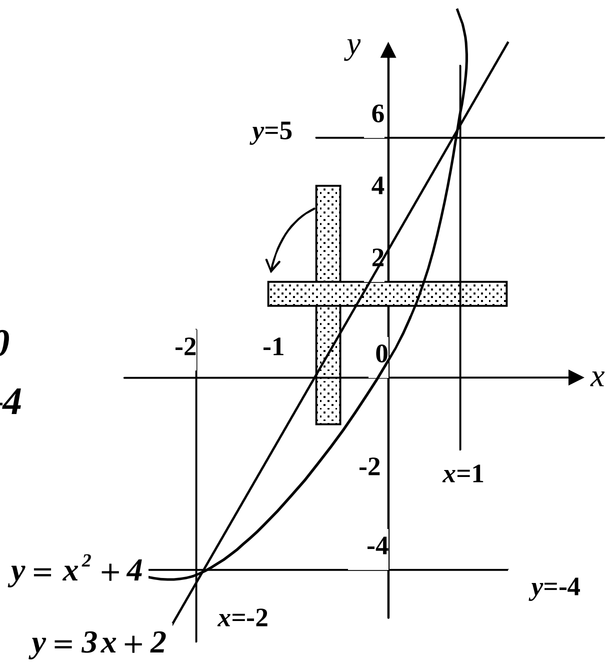
Sol.-

$$1)\quad \left.\begin{array}{l} y = 3x + 2 \ \ .....(1) \\ y = x^2 + 4x \ \ ...(2) \end{array}\right\} \Rightarrow$$

$$(x+2)(x-1) = 0$$

$$\text{either } x = -2 \Rightarrow y = -4$$

$$\text{or } \quad x = 1 \Rightarrow y = 5$$



$$(a)\int_{-2}^{1}\int_{x^2+4x}^{3x+2} dy\,dx = \int_{-2}^{1} y\,\Big|_{x^2+4x}^{3x+2} dx = \int_{-2}^{1}(2-x-x^2)dx$$

$$= 2x-\frac{x^2}{2}-\frac{x^3}{3}\Big|_{-2}^{1} = 2-\frac{1}{2}-\frac{1}{3}-(-4-2+\frac{8}{3}) = \frac{9}{2}$$

(b) *The reversed integral is :-*

$$y = 3x+2 \quad \Rightarrow \quad x = \frac{y-2}{3}$$

$$y = x^2+4x \Rightarrow (x+2)^2 = y+4 \Rightarrow x = -2 \mp \sqrt{y+4}$$

$$Since\ -2 \le x \le 1 \quad \Rightarrow \quad x = -2+\sqrt{y+4}$$

$$\int_{-4}^{5}\int_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} dx\,dy = \int_{-4}^{5} x\Big|_{\frac{y-2}{3}}^{-2+\sqrt{y+4}} = \int_{-4}^{5}\left(-2+\sqrt{y+4}-\frac{y-2}{3}\right)dy$$

$$= -2y+\frac{2}{3}(y+4)^{3/2}-\frac{(y-2)^2}{6}\Big|_{-4}^{5}$$

$$= -10+\frac{2}{3}(27)-\frac{9}{6}-(8+0-\frac{36}{6}) = \frac{9}{2}$$

$=$ *The same result as in (a).*

$$2)\ (a)\int_{-1}^{0}\int_{-2x}^{1-x} dy\,dx + \int_{0}^{2}\int_{-\frac{x}{2}}^{1-x} dy\,dx = \int_{-1}^{0} y\,\Big|_{-2x}^{1-x} dx + \int_{0}^{2} y\,\Big|_{-\frac{x}{2}}^{1-x} dx$$

$$= \int_{-1}^{0}(1+x)\,dx + \int_{0}^{2}(1-\frac{x}{2})\,dx = x+\frac{x^2}{2}\Big|_{-1}^{0} + x-\frac{x^2}{4}\Big|_{0}^{2}$$

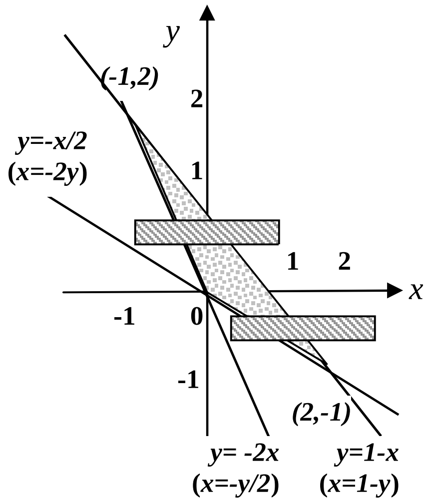
$$= 0-(-1+\frac{1}{2})+2-1-0 = \frac{3}{2}$$

(b) 1st region

$$\left.\begin{aligned} y &= 1-x\ .....(1)\\ y &= -2x\ ......(2)\end{aligned}\right\} \Rightarrow x=-1 \Rightarrow y=2 \qquad x\ \ from\ -1\ \ to\ \ 0$$

2nd region

$$\left.\begin{aligned} y &= 1 - x \quad ...(1) \\ y &= -\frac{x}{2} \quad .....(2) \end{aligned}\right\} \Rightarrow x = 2 \Rightarrow y = -1 \qquad y \ \ from\ \ 0\ \ to\ \ 2$$



$$\int_0^2 \int_{-\frac{y}{2}}^{1-y} dx\,dy + \int_{-1}^{0} \int_{-2y}^{1-y} dx\,dy = \int_0^2 x \Big|_{-\frac{y}{2}}^{1-y} dy + \int_{-1}^{0} x \Big|_{-2y}^{1-y} dy$$

$$= \int_0^2 (1 - \frac{y}{2})dy + \int_{-1}^{0} (1 + y)dy = y - \frac{y^2}{4}\Big|_0^2 + y + \frac{y^2}{2}\Big|_{-1}^{0}$$

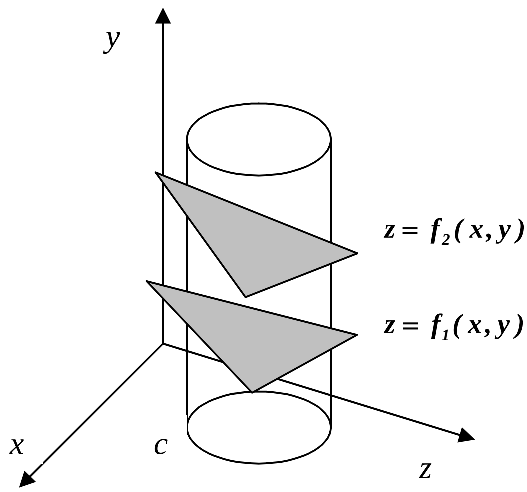
$$= 2 - 1 - 0 + 0 - (-1 + \frac{1}{2}) = \frac{3}{2}$$

$= The\ same\ result\ as\ in\ \ (a).$

7-3- <u>Triple integrals (Volume)</u>:

Consider a region N in *xyz*-space bounded below by a surface $z = f_1(x,y)$, above by the surface $z = f_2(x,y)$ and laterally by a cylinder c with elements parallel to the *z*-axis. Let A denote the region of the *xy*-plane enclosed by cylinder c (that is, A is the region covered by the orthogonal projection of the solid into *xy*-plane). Then the volume V of the region V can be found by evaluating the triply iterated integral:-

$$V = \iint_A \int_{f_1(x,y)}^{f_2(x,y)} dz\,dy\,dx$$



Let *z*-limits of integration indicate that for every (*x*,*y*) in the region *A*,*Z* may extend from the lower surface $z = f_1(x,y)$ to the surface $z = f_2(x,y)$. The *y*- and *x*-limits of integration have not been given explicitly in equation above, but are indicated as extending over the region *A*.

We can find the equation of the boundary of the region *A* by eliminating z between the two equations $z = f_1(x,y)$ and $z = f_2(x,y)$, thus obtaining an equation $f_1(x,y) = f_2(x,y)$ which contains no *z*, and interpret it as an equation in the *xy*-plane.

EX-9 **The volume in the first octant bounded by the cylinder $x = 4 - y^2$, and the planes $z = y,\ x = 0,\ z = 0.$**

***Sol.*-**

$$x = 4 - y^2 \quad \Rightarrow \quad y = \mp\sqrt{4 - x} \qquad in\ first\ octant: -$$

$$V = \int_0^4 \int_0^{\sqrt{4-x}} \int_0^y dz\,dy\,dx = \int_0^4 \int_0^{\sqrt{4-x}} z\Big|_0^y dy\,dx = \int_0^4 \int_0^{\sqrt{4-x}} y\,dy\,dx = \int_0^4 \frac{y^2}{2}\Big|_0^{\sqrt{4-x}} dx$$

$$= \frac{1}{2}\int_0^4 (4 - x - 0)dx = \frac{1}{2}\left[4x - \frac{x^2}{2}\right]_0^4 = \frac{1}{2}\left[16 - \frac{16}{2} - 0\right] = 4$$

EX-10 **The volume enclosed by the cylinders $z = 5 - x^2, z = 4x^2$ and the planes $y = 0,\ \ x + y = 1$.**

***Sol.*-**

$$\left.\begin{aligned} z &= 5 - x^2 \ ...(1) \\ z &= 4x^2 \ ......(2) \end{aligned}\right\} \Rightarrow \ x = \mp 1$$

$$V = \int_{-1}^{1} \int_0^{1-x} \int_{4x^2}^{5-x^2} dz\,dy\,dx = \int_{-1}^{1} \int_0^{1-x} z\Big|_{4x^2}^{5-x^2} dy\,dx = \int_{-1}^{1} \int_0^{1-x} (5 - 5x^2)\,dy\,dx$$

$$= 5\int_{-1}^{1} (1 - x^2)\,y\Big|_0^{1-x} dx = 5\int_{-1}^{1} (1 - x^2)(1 - x)dx$$

$$= 5\int_{-1}^{1} (1 - x - x^2 + x^3)\,dx = 5\left[x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4}\right]_{-1}^{1}$$

$$= 5\left[(1+1) - \frac{1}{2}(1-1) - \frac{1}{3}(1+1) + \frac{1}{4}(1-1)\right] = \frac{20}{3}$$

EX-11 **The volume enclosed by the cylinders** $y^2 + 4z^2 = 16$ **and the planes** $x = 0$, $x + y = 4$.

Sol.-

$$y^2 + 4z^2 = 16 \quad \Rightarrow \quad y = \mp 2\sqrt{4 - z^2}$$

$$V = \int_{-2}^{2} \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} \int_{0}^{4-y} dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} (4 - y)\, dy\, dz = \int_{-2}^{2} 4y - \frac{y^2}{2} \Bigg|_{-2\sqrt{4-z^2}}^{2\sqrt{4-z^2}} dz = 16 \int_{-2}^{2} \left(4 - z^2\right)^{1/2} dz$$

$$let \quad z = 2\sin\theta \;\Rightarrow\; dz = 2\cos\theta\, d\theta \;,\quad \theta = \sin^{-1}\frac{z}{2} \quad \overset{at\; z=2 \;\Rightarrow\; \theta = \frac{\pi}{2}}{\underset{at\; z=2 \;\Rightarrow\; \theta = \frac{\pi}{2}}{\Rightarrow\Rightarrow\Rightarrow}}$$

$$V = 16 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (4 - 4\sin^2\theta)^{1/2}\, 2\cos\theta\, d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta = 64 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= 32 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 32 \left[\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + \frac{1}{2}(0 - 0) \right] = 32\pi$$

EX-12 **The volume bounded by the ellipse paraboloids** $z = x^2 + 9y^2$ **and** $z = 18 - x^2 - 9y^2$.

Sol.-

$$\left.\begin{aligned} z &= 18 - x^2 - 9y^2 \;\; ..(1) \\ z &= x^2 + 9y^2 \;\; ....(2) \end{aligned}\right\} \Rightarrow \; 9 - x^2 - 9y^2 = 0 \;\Rightarrow\; y = \mp \frac{1}{3}\sqrt{9 - x^2}$$

$$V = \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} \int_{x^2+9y^2}^{18-x^2-9y^2} dz\, dy\, dx = \int_{-3}^{3} \int_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}} \left[18 - x^2 - 9y^2 - (x^2 + 9y^2) \right] dy\, dx$$

$$V=2\int_{-3}^{3}(9-x^2)y-3y^3\Big]_{-\frac{1}{3}\sqrt{9-x^2}}^{\frac{1}{3}\sqrt{9-x^2}}dx$$

$$=2\int_{-3}^{3}\left[(9-x^2)\left(\frac{\sqrt{9-x^2}}{3}+\frac{\sqrt{9-x^2}}{3}\right)-3\left(\frac{(9-x^2)^{3/2}}{27}+\frac{(9-x^2)^{3/2}}{27}\right)\right]dx$$

$$=\frac{8}{9}\int_{-3}^{3}(9-x^2)^{3/2}dx$$

$$let\ \ x=3sin\theta\ \Rightarrow\ dx=3cos\theta\,d\theta\quad,\quad \theta=sin^{-1}\frac{x}{3}\ \overset{at\ x=3\ \Rightarrow\ \theta=\frac{\pi}{2}}{\underset{at\ x=-3\ \Rightarrow\ \theta=-\frac{\pi}{2}}{\Rightarrow\Rightarrow\Rightarrow}}$$

$$=\frac{8}{9}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(9-9\sin^2\theta)^{3/2}\,3cos\theta\,d\theta=72\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}cos^4\theta\,d\theta=72\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+cos2\theta}{2}\right)^2d\theta$$

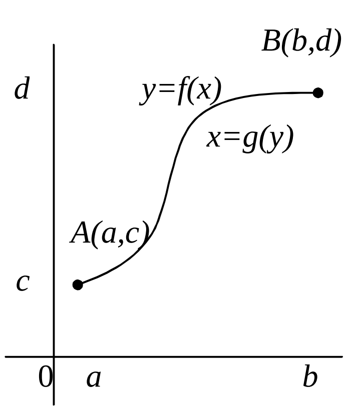
$$=18\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2cos2\theta+cos^2 2\theta)\,d\theta=18\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2cos2\theta+\frac{cos4\theta}{2})\,d\theta$$

$$=9\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(3+4cos2\theta+cos4\theta)\,d\theta=9\left[3\theta+2\sin2\theta+\frac{1}{4}\sin4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=9\left[3(\frac{\pi}{2}+\frac{\pi}{2})+2(\sin\pi-\sin(-\pi))+\frac{1}{4}(\sin2\pi-\sin(-2\pi))\right]=27\pi$$

7-4- <u>The length of a plane curve:-</u>

The length of the curve $y=f(x)$ from point *A(a,c)* to *B(b,d)* is:-



$$L=\int_a^b\sqrt{1+(\frac{dy}{dx})^2}\,dx$$

If x can be expressed as a function of y then the length is:-

$$L=\int_c^d\sqrt{1+(\frac{dx}{dy})^2}\,dy$$

Let the equation of motion be $x = g(t)$ and $y = h(t)$ continuously differentiable for t between $t_a(at\ A)$ and $t_b(at\ B)$, then the length of the curve is:-

$$L = \int_{t_a}^{t_b} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\ dt$$

EX-13 – **Find the length of the curve:**

1) $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}}$ from $x = 0$ to $x = 3$

2) $9x^2 = 4y^3$ from $(0,0)$ to $(2\sqrt{3},3)$

3) $y = x^{\frac{2}{3}}$ from $x = -1$ to $x = 8$

Sol.

1) $y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}} \Rightarrow \frac{dy}{dx} = x(x^2 + 2)^{\frac{1}{2}}$

$$L = \int_0^3 \sqrt{1 + x^2(x^2 + 2)}\ dx = \int_0^3 (x^2 + 1)\ dx = \frac{x^3}{3} + x \Big|_0^3 = 9 + 3 - 0 = 12$$

2) $9x^2 = 4y^3 \Rightarrow x = \mp\frac{2}{3}y^{\frac{3}{2}}$ Since x from 0 to $2\sqrt{3}$

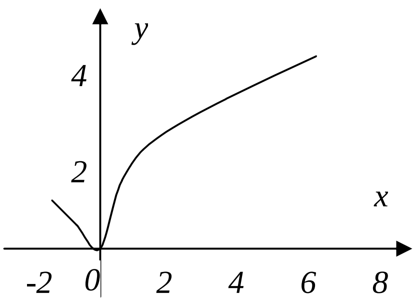
then $x = \frac{2}{3}y^{\frac{3}{2}} \Rightarrow \frac{dx}{dy} = y^{\frac{1}{2}}$

$$L = \int_0^3 \sqrt{1 + y}\ dy = \frac{2}{3}(1 + y)^{\frac{3}{2}}\Big|_0^3 = \frac{2}{3}[8 - 1] = \frac{14}{3}$$

3) $y = x^{\frac{2}{3}} \Rightarrow \frac{dy}{dx} = \frac{2}{3}x^{-\frac{1}{3}}$

Since $\frac{dy}{dx} = \infty$ at $x = 0$

then $x = \mp y^{\frac{3}{2}} \Rightarrow \frac{dx}{dy} = \mp\frac{3}{2}y^{\frac{1}{2}}$



$$L = \int_0^1 \sqrt{1 + \frac{9}{4}y}\ dy + \int_0^4 \sqrt{1 + \frac{9}{4}y}\ dy = \frac{1}{18}\left[\frac{(4 + 9y)^{\frac{3}{2}}}{3/2}\Big|_0^1 + \frac{(4 + 9y)^{\frac{3}{2}}}{3/2}\Big|_0^4\right]$$

$$= \frac{1}{27}\left[(13\sqrt{13} - 4\sqrt{4}) + (40\sqrt{40} - 4\sqrt{4})\right] = 10.51$$

EX-14 – **Find the distance traveled between** $t=0$ **and** $t=\frac{\pi}{2}$ **a particle** $P(x,y)$ **whose position at time** t **is given by:-** $x=a\cos t+a\cdot t\sin t$ *and* $y=a\sin t-a\cdot t\cos t$ **where** a **is a positive constant.**

Sol.

$$x=a\cos t+a\cdot t\sin t \;\Rightarrow\; \frac{dx}{dt}=a\cdot t\cos t$$

$$y=a\sin t-a\cdot t\cos t \;\Rightarrow\; \frac{dy}{dt}=a\cdot t\sin t$$

$$L=\int_a^b\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=\int_0^{\frac{\pi}{2}}\sqrt{a^2\cdot t^2\cos^2 t+a^2\cdot t^2\sin^2 t}\,dt$$

$$=a\int_0^{\frac{\pi}{2}}t\,dt=\frac{a}{2}t^2\Big|_0^{\frac{\pi}{2}}=\frac{a}{2}\left[\frac{\pi^2}{4}-0\right]=\frac{a}{8}\pi^2$$

EX-15 – **Find the length of the curve:-**

$$x=t-\sin t \quad and \quad y=1-\cos t\;;\;0\le t\le 2\pi$$

Sol.

$$x=t-\sin t \;\Rightarrow\; \frac{dx}{dt}=1-\cos t$$

$$y=1-\cos t \;\Rightarrow\; \frac{dy}{dt}=\sin t$$

$$L=\int_a^b\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt=\int_0^{2\pi}\sqrt{(1-\cos t)^2+\sin^2 t}\,dt$$

$$=\int_0^{2\pi}\sqrt{1-2\cos t+\cos^2 t+\sin^2 t}\,dt=\int_0^{2\pi}\sqrt{1-2\cos t+1}\,dt$$

$$=2\int_0^{2\pi}\sqrt{\frac{1-\cos t}{2}}\,dt=2\int_0^{2\pi}\sin\frac{t}{2}\,dt=-4\cos\frac{t}{2}\Big|_0^{2\pi}$$

$$=-4\left[\cos\pi-\cos 0\right]=-4\left[-1-1\right]=8$$

7-5- **<u>The surface area</u>:**

Suppose that the curve $y = f(x)$ is rotated about the *x*-axis. It will generate a surface in space. Then the surface area of the shape is:-

$$S = \int_a^b 2\pi\, y \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

If the curve rotated about the *y*-axis, then the surface area is:-

$$S = \int_c^d 2\pi\, x \sqrt{1 + (\frac{dx}{dy})^2}\, dy$$

If the curve sweeps out the surface is given in parametric form with x and y as functions of a third variable t that varies from t_a to t_b then we may compute the surface area from the formula:-

$$S = \int_{t_a}^{t_b} 2\pi\, \rho \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$$

where ρ is the distance from the axis of revolution to the element of arc length and is expressed as a function of *t*.

<u>EX-16</u> **– The circle $x^2 + y^2 = r^2$ is revolved about the *x*-axis. Find the area of the sphere generated.**

***<u>Sol.</u>*-**

$$y = \sqrt{r^2 - x^2} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{\sqrt{r^2 - x^2}}$$

$$S = \int_a^b 2\pi\, y \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_{-r}^{r} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = 2\pi\, r \int_{-r}^{r} dx$$

$$= 2\pi\, r\, x \Big|_{-r}^{r} = 2\pi\, r\big(r - (-r)\big) = 4\pi\, r^2$$

<u>*EX-17*</u> – Find the area of the surface generated by rotating the portion of the curve $y=\frac{1}{3}(x^2+2)^{\frac{3}{2}}$ between *x=0* and *x=3* about the *y*-axis.

<u>*Sol.*</u>-

$$y=\frac{1}{3}(x^2+2)^{\frac{3}{2}} \Rightarrow x=((3y)^{\frac{2}{3}}-2)^{\frac{1}{2}} \Rightarrow \frac{dx}{dy}=\frac{1}{(3y)^{\frac{1}{3}}\cdot((3y)^{\frac{2}{3}}-2)^{\frac{1}{2}}}$$

$$y=\frac{1}{3}(x^2+2)^{\frac{3}{2}} \quad \overset{at\ x=0}{\Rightarrow\Rightarrow\Rightarrow} \quad y=\frac{2\sqrt{2}}{3} \quad and \quad \overset{at\ x=3}{\Rightarrow\Rightarrow\Rightarrow} \quad y=\frac{11\sqrt{11}}{3}$$

$$S=\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}} 2\pi\sqrt{(3y)^{\frac{2}{3}}-2}\cdot\sqrt{1+\frac{1}{((3y)^{\frac{2}{3}}-2)(3y)^{\frac{2}{3}}}}\,dy$$

$$=2\pi\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}\sqrt{\frac{(3y)^{\frac{4}{3}}-2(3y)^{\frac{2}{3}}+1}{(3y)^{\frac{2}{3}}}}\,dy=2\pi\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}\sqrt{\frac{((3y)^{\frac{2}{3}}-1)^2}{(3y)^{\frac{2}{3}}}}\,dy$$

$$=2\pi\int_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}\left[(3y)^{\frac{1}{3}}-(3y)^{-\frac{1}{3}}\right]dy=2\pi\left[\frac{1}{3}\frac{(3y)^{\frac{4}{3}}}{\frac{4}{3}}-\frac{1}{3}\frac{(3y)^{\frac{2}{3}}}{\frac{2}{3}}\right]_{\frac{2\sqrt{2}}{3}}^{\frac{11\sqrt{11}}{3}}$$

$$=\pi\left[\frac{(3\cdot\frac{11\sqrt{11}}{3})^{\frac{4}{3}}}{2}-(3\cdot\frac{11\sqrt{11}}{3})^{\frac{2}{3}}-\frac{(3\cdot\frac{2\sqrt{2}}{3})^{\frac{4}{3}}}{2}-(3\cdot\frac{2\sqrt{2}}{3})^{\frac{2}{3}}\right]=\frac{99}{2}\pi$$

<u>*EX-18*</u> – The arc of the curve $y=\frac{x^3}{3}+\frac{1}{4x}$ from *x=1* to *x=3* is rotated about the line *y= -1*. Find the surface area generated.

<u>*Sol.*</u>-

$$y = \frac{x^3}{3} + \frac{1}{4x} \quad \Rightarrow \quad \frac{dy}{dx} = x^2 - \frac{1}{4x^2} = \frac{4x^4 - 1}{4x^2}$$

$$S = 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\sqrt{1 + \frac{(4x^4 - 1)^2}{16x^4}}\, dx$$

$$= 2\pi \int_1^3 \frac{4x^4 + 12x + 3}{12x} \sqrt{\frac{(4x^4 + 1)^2}{16x^4}}\, dx$$

$$= \frac{\pi}{24} \int_1^3 (16x^5 + 48x^2 + 16x + 12x^{-2} + 3x^{-3})dx$$

$$= \frac{\pi}{24} \left[\frac{8}{3}x^6 + 16x^3 + 8x^2 - \frac{12}{x} - \frac{3}{2x^2}\right]_1^3$$

$$= \frac{\pi}{24} \left[\frac{8}{3}(729 - 1) + 16(27 - 1) + 8(9 - 1) - 12(\frac{1}{3} - 1) - \frac{3}{2}(\frac{1}{9} - 1)\right]$$

$$= \frac{1823}{18}\pi$$

EX-19 **– Find the area of the surface generated by rotating the curve** $x = t^2 \quad , \quad y = t \quad , \quad 0 \le t \le 1$ **about the** x**-axis.**

Sol.-

$$x = t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2t \quad \text{and} \quad y = t \quad \Rightarrow \quad \frac{dy}{dt} = 1$$

$$S = \int_{t_a}^{t_b} 2\pi\, \rho \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt = 2\pi \int_0^1 t\sqrt{4t^2 + 1}\, dt$$

$$= \frac{\pi}{4}\left[\frac{(4t^2 + 1)^{\frac{3}{2}}}{3/2}\right]_0^1 = \frac{\pi}{6}\left[5\sqrt{5} - 1\right]$$

Problems – 7

1) **Find the area of the region bounded by the given curves and lines for the following problems:-**

1. **The coordinate axes and the line** $x + y = a$
2. **The** x**-axis and the curve** $y = e^x$ **and the lines** $x = 0$, $x = 1$
3. **The curve** $y^2 + x = 0$ **and the line** $y = x + 2$
4. **The curves** $x = y^2$ **and** $x = 2y - y^2$
5. **The parabola** $x = y - y^2$ **and the line** $x + y = 0$

$$(ans.:\ 1.\frac{a^2}{2}\ ;2.e-1\ ;3.\frac{9}{2}\ ;4.\frac{1}{3}\ ;5.\frac{4}{3})$$

2) **Write an equivalent double integral with order of integration reversed for each integrals check your answer by evaluation both double integrals, and sketch the region.**

1. $\int_0^2 \int_1^{e^x} dy\, dx$ $\qquad (ans.:\ \int_1^{e^2} \int_{\ln y}^{2} dx\, dy\ ;\ e^2 - 3)$

2. $\int_0^1 \int_{\sqrt{y}}^{1} dx\, dy$ $\qquad (ans.:\ \int_0^1 \int_0^{x^2} dy\, dx\ ;\ \frac{1}{3})$

3. $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\, dx\, dy$ $\qquad (ans.:\ \int_{-2}^{2} \int_0^{\sqrt{\frac{4-x^2}{2}}} y\, dy\, dx\ ;\ \frac{8}{3})$

3) **Find the volume of the tetrahedron bounded by the plane** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **and the coordinate planes.**

$$(ans.:\ \frac{1}{6}|abc|)$$

4) **Find the volume bounded by the plane** $z = 0$ **laterally by the elliptic cylinder** $x^2 + 4y^2 = 4$ **and above by the plane** $z = x + 2$ **.**

$$(ans.:\ 4\pi)$$

5) **Find the lengths of the following curves:-**

1. $y = x^{\frac{3}{2}}$ *from* $(0,0)$ *to* $(4,8)$ $\quad$ $(ans.:\ \frac{8}{27}(10\sqrt{10}-1))$

2. $y = \frac{x^3}{3} + \frac{1}{4x}$ *from* $x = 1$ *to* $x = 3$ $\quad$ $(ans.:\ \frac{53}{6})$

3. $x = \frac{y^4}{4} + \frac{1}{8y^2}$ *from* $y = 1$ *to* $y = 2$ $\quad$ $(ans.:\ \frac{123}{32})$

4. $(y+1)^2 = 4x^3$ *from* $x = 0$ *to* $x = 1$ $\quad$ $(ans.:\ \frac{4}{27}(10\sqrt{10}-1))$

6) **Find the distance traveled by the particle *P(x,y)* between *t=0* and *t=4* if the position at time *t* is given by:** $x = \frac{t^2}{2}$; $y = \frac{1}{3}(2t+1)^{\frac{3}{2}}$

$(ans.:\ 12)$

7) **The position of a particle *P(x,y)* at time *t* is given by:** $x = \frac{1}{3}(2t+3)^{\frac{3}{2}}$; $y = \frac{t^2}{2} + t$ **. Find the distance it travel between *t=0* and *t=3*.** $\quad$ $(ans.:\ \frac{21}{2})$

8) **Find the area of the surface generated by rotating about the *x*-axis the arc of the curve** $y = x^3$ *between* $x = 0$ *and* $x = 1$.

$(ans.:\ \frac{\pi}{27}(10\sqrt{10}-1))$

9) **Find the area of the surface generated by rotating about the *y*-axis the arc of the curve** $y = x^2$ *between* $(0,0)$ *and* $(2,4)$.

$(ans.:\ \frac{\pi}{6}(17\sqrt{17}-1))$

10) **Find the area of the surface generated by rotating about the *y*-axis the curve** $y = \frac{x^2}{2} + \frac{1}{2}$; $0 \le x \le 1$. $\quad$ $(ans.:\ \frac{2}{3}\pi(2\sqrt{2}-1))$

11) **The curve described by the particle *P(x,y)*** $x = t+1$, $y = \frac{t^2}{2} + t$ **from** $t = 0$ **to** $t = 4$ **is rotated about the *y*-axis. Find the surface area that is generated.**

$(ans.:\ \frac{2\sqrt{2}}{3}\pi(13\sqrt{13}-1))$

Chapter eight

Matrices and Determinants

A matrix is a rectangular array of elements (scalars) from a field. The order, or size, of a matrix is specified by the number of rows and the number of columns, i.e. A an "m by n" matrix has m rows and n columns, and the element in the i th row and j th column is often denoted by a_{ij} :

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}$$

A vector is a matrix with a single row (or column) of n elements , i.e. the column vector is:-

$$A = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} \quad \textit{and row vector is} \quad A = \left[a_1 \quad a_2 \quad . \quad . \quad a_n\right]$$

The matrix is square if the number of rows and columns are equal (i.e. $m = n$) and the elements a_{ij} of a square matrix are called the main diagonal.

The identity matrix: $I = \begin{bmatrix} 1 & 0 & .... & 0 \\ 0 & 1 & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ 0 & 0 & .... & 1 \end{bmatrix}$ **is square matrix with one in each main diagonal position and zeros else.**

The diagonal matrix $D=\begin{bmatrix} a_1 & 0 & .... & 0 \\ 0 & a_2 & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ 0 & 0 & .... & a_n \end{bmatrix}$ **has the elements** $a_1, a_2, \ldots, a_n$ **in its main diagonal position and zeros in all other locations, some of the** a_i **may be zero but not all.**

A $n \times n$ **triangular matrix has the pattern:-**

$$\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ 0 & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ 0 & 0 & .... & a_{nn} \end{bmatrix} \quad or \quad \begin{bmatrix} a_{11} & 0 & .... & 0 \\ a_{21} & a_{22} & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$$

lower triangular matrix *upper triangular matrix*

The $m \times n$ **null matrix:-** $0=\begin{bmatrix} 0 & 0 & .... & 0 \\ 0 & 0 & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ 0 & 0 & .... & 0 \end{bmatrix}$ **has zero in each of its positions.**

Elementary operations with matrices and vectors

1. Equality:- Two $m \times n$ **matrices and** A **and** B **are said to be equal if:** $a_{ij}=b_{ij} \quad \forall \text{ pairs of } i \text{ and } j.$

EX-1 **– Find the values of** x, y **for the following matrix equation:**

$$\begin{bmatrix} x-2y & 0 \\ -2 & 6 \end{bmatrix}=\begin{bmatrix} 3 & 0 \\ -2 & x+y \end{bmatrix}$$

Sol. **–**

$$\left.\begin{array}{ll} x-2y=3 & ....(1) \\ x+y=6 & ....(2)*2 \end{array}\right\} \Rightarrow \begin{array}{c} x-2y=3 \\ \underline{2x+2y=12} \\ 3x=15 \Rightarrow \boxed{x=5} \end{array}$$

$$substitution\ \ x=5\ \ in(2)\ \Rightarrow\ 5+y=6\ \Rightarrow\ \boxed{y=1}$$

2. <u>Addition:-</u> The sum of two matrices of like dimensions is the matrix of the sum of the corresponding elements. If:-

$$A=\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} \quad , \quad B=\begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ b_{21} & b_{22} & .... & b_{2n} \\ .... & .... & .... & .... \\ b_{m1} & b_{m2} & .... & b_{mn} \end{bmatrix}$$

then

$$A \mp B=\begin{bmatrix} a_{11}\mp b_{11} & a_{12}\mp b_{12} & .... & a_{1n}\mp b_{1n} \\ a_{21}\mp b_{21} & a_{22}\mp b_{22} & .... & a_{2n}\mp b_{2n} \\ .... & .... & .... & .... \\ a_{m1}\mp b_{m1} & a_{m2}\mp b_{m2} & .... & a_{mn}\mp b_{mn} \end{bmatrix}$$

thus:

1) ***A+B = B+A***
2) ***A+(B+C) = (A+B)+C***
3) ***A-(B-C) = A-B+C***

***<u>EX-2</u>*- Find *A+B* and *A-B* if:-**

$$A=\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \quad , \quad B=\begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

<u>Sol.</u>-

$$A+B=\begin{bmatrix} 2+1 & 1-2 & 3+2 \\ 1+2 & 0+3 & -2-1 \end{bmatrix}=\begin{bmatrix} 3 & -1 & 5 \\ 3 & 3 & -3 \end{bmatrix}$$

$$A-B=\begin{bmatrix} 2-1 & 1-(-2) & 3-2 \\ 1-2 & 0-(+3) & -2-(-1) \end{bmatrix}=\begin{bmatrix} 1 & 3 & 1 \\ -1 & -3 & -1 \end{bmatrix}$$

3. <u>Multiplication by a scalar:-</u> The matrix A is multiplied by the scalar C by multiplying each element of A by c:-

$$CA = \begin{bmatrix} ca_{11} & ca_{12} & .... & ca_{1n} \\ ca_{21} & ca_{22} & .... & ca_{2n} \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ ca_{m1} & ca_{m2} & .... & ca_{mn} \end{bmatrix}$$

<u>EX-3</u>- Assume $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & -1 \end{bmatrix}$, find $3A$.

<u>Sol.</u>-

$$3A = \begin{bmatrix} 3*3 & 3*2 & 3*1 \\ 3*0 & 3*5 & 3*(-1) \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 0 & 15 & -3 \end{bmatrix}$$

4. <u>Matrix multiplication:-</u> For the matrix product AB to be defined it is necessary that the number of columns of A be equal to the number of rows of B. The dimensions of such matrices are said to be conformable. If A is of dimensions $m \times p$ and B is $p \times n$, then the $ij\ th$ element of the product $C=AB$ is computed as:-

$$C_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

This is the sum of the products of corresponding elements in the $i\ th$ row of A and $j\ th$ column of B. The dimensions of AB are of course $m \times n$.

<u>EX-4</u>- Assume $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 & 5 & 4 \\ -1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ find AB.

<u>Sol.</u>-

$$AB = \begin{bmatrix} 1*6+2(-1)+3*0 & 1*5+2*1+3*2 & 1*4+2(-1)+3*0 \\ -1*6+0(-1)+1*0 & -1*5+0*1+1*2 & -1*4+0(-1)+1*0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 13 & 2 \\ -6 & -3 & -4 \end{bmatrix}$$

Properties of multiplication:-

$a)\ A(B+C) = AB + AC$ *distributive law*

$b)\ A(BC) = (AB)C$ *associative law*

$c)\ AB \neq BA$ *commutative law does not hold*

$d)\ AI = IA = A$

EX-5- Assume $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, verify that $AB \neq BA$.

Sol.-

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 6 & 3 \end{bmatrix} \quad \& \quad BA = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 2 & 7 \end{bmatrix}$$

Hence $AB \neq BA$

5. Transpose of matrix:- **Let A is any $m \times n$ matrix the transpose of A is $n \times m$ matrix A' formed by interchanging the role of rows and columns.**

$$A' = \begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{21} & .... & a_{m1} \\ a_{12} & a_{22} & .... & a_{m2} \\ .... & .... & .... & .... \\ a_{1n} & a_{2n} & .... & a_{mn} \end{bmatrix}$$

If a matrix is square and equal to its transpose, it is said to be symmetric, then $a_{ij} = a_{ji}$ for all pairs of i and j.

Properties of transpose are:-

$a)\ (A \mp B)' = A' \mp B'$

$b)\ (AB)' = B'A'$

$c)\ (A')' = A$

EX-6- Assume $A = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & -1 & 0 \\ 5 & 4 & 3 \\ 2 & 1 & -1 \end{bmatrix}$, show that:-

$1)\ A$ *is symmetric matrix* $\qquad 2)\ (A+B)' = A' + B'$

$3)\ (AB)' = B'A'$

Sol.-

$$1)\ A' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = A \Rightarrow A \text{ is a symmetric matrix.}$$

$$2)\ L.H.S. = (A+B)' = \begin{bmatrix} 7 & 1 & 5 \\ 7 & 3 & 7 \\ 7 & 5 & -1 \end{bmatrix}' = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix}$$

$$R.H.S. = A' + B' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix} = L.H.S.$$

$$\therefore (A+B)' = A' + B'$$

$$3)\ L.H.S. = (AB)' = \begin{bmatrix} 32 & 10 & 1 \\ 11 & -2 & -7 \\ 40 & 11 & 12 \end{bmatrix}' = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix}$$

$$R.H.S. = B'A' = \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix} = L.H.S.$$

$$\therefore (AB)' = B'A'$$

6. <u>Vector inner product:-</u> The inner product of two vectors with the same number of elements is defined to be the sum of the products of the corresponding elements:-

$$A'B = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i \, b_i$$

Since the inner product is a scalar, hence $A'B = B'A$. Moreover, the inner product of two vectors may be taken the following term:-

$$A\,B' = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \cdots & \cdots & \cdots & \cdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix}$$

Which is $n \times n$ *matrix.*

EX-7- Let $A = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, find $A'B$ and AB'

Sol.-

$$A'B = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 5 * 2 + (-2) * (-1) + 1 * 3 = 15$$

$$AB' = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 15 \\ -4 & 2 & -6 \\ 2 & -1 & 3 \end{bmatrix}$$

Determinants

The minor of the element a_{ij} in a matrix A is the determinant of the matrix that remains when the row and column containing a_{ij} are deleted. For example, let:-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then the minor of } a_{21} \text{ is } \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \text{ then the minor of } a_{34} \text{ is } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

and so on.

The cofactor of a_{ij} is the determinant A_{ij} that is $(-1)^{i+j}$ times the minor of a_{ij} . Thus:-

$$for\ matrix\ (3\times 3) \Rightarrow A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$for\ matrix\ (4\times 4) \Rightarrow A_{31} = (-1)^{3+1}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

With each square matrix A we associate a number $det\ A$ or $|A|$ or $|a_{ij}|$ called the determinant of A, calculated from the entries of A in the following way:-

$$for\ \ n=1\ ,\ A=[a] \Rightarrow |A| = a$$

$$for\ \ n=2\ ,\ A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}\ a_{22} - a_{12}\ a_{21}$$

$$for\ \ n=3\ ,\ A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

The determinant of a square matrix can be calculated from the cofactors of any row or any column.

***EX-8*- Find the determinant of the matrix:-** $A=\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$

Sol.-

1^st^ method

$$A=\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}\begin{matrix} 2 & 1 \\ 3 & -1 \\ 2 & 3 \end{matrix}$$

$$= 2(-1)\cdot 1 + 1(-2)\cdot 2 + 3\cdot 3\cdot 3 - (3(-1)\cdot 2 + 2(-2)\cdot 3 + 1\cdot 3\cdot 1)$$

$$= 36$$

2nd method

If we were to expand the determinant by cofactors according to elements of its third column, say, we would get:-

$$A = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

$$= 3(-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + (-2)(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 1(-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 3(9-(-2)) + 2(6-2) + (-2-3) = 36$$

Useful facts about determinants:-

F-1: If two rows of matrix are identical, the determinant is zero.

EX-9 **Show that:-** $\begin{vmatrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$

Sol.-

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{vmatrix}\begin{matrix} 3 & -1 \\ 2 & -3 \\ 3 & -1 \end{matrix} \quad = -18-15-4-(-18-15-4) = 0$$

F-2: Interchanging two rows of matrix changes the sign of its determinants.

EX-10 **Show that:-** $\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & 0 \\ 1 & -2 \end{matrix} \quad = 0+3+10-(0-12-4) = 29$$

$$R.H.S. = \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}\begin{matrix} -1 & 0 \\ 2 & 1 \\ 1 & -2 \end{matrix} \quad = -(-4+0-12-(3+10+0)) = 29 = L.H.S.$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$$

F-3: The determinant of the transpose of a matrix is equal to the original determinant.

EX-11 **Show that:-** $\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = 29 \quad from\ ex\text{-}10$$

$$R.H.S. = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 0 \\ 5 & 3 \end{matrix} = 0 + 10 + 3 - (0 - 12 - 4) = 29 \quad = \quad L.H.S.$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$$

F-4: If each element of same row (or column) of a matrix is multiplied by a constant *C*, the determinant is multiplied by *C*.

EX-12 **Show that:-** $\begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = \begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} \begin{matrix} 6 & 3 \\ -1 & 0 \\ 1 & -2 \end{matrix} = 0 + 9 + 30 - (0 - 36 - 12) = 87$$

$$R.H.S. = 3 * 29 = 87 = L.H.S.$$

$$\therefore \begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$$

F-5: If all elements of a matrix above the main diagonal (or all below it) are zero, the determinant of the matrix is the product of the elements on the main diagonal.

EX-13 **Find:-** $\begin{vmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{vmatrix}$

Sol.-

$$\left|\begin{matrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{matrix}\right|\begin{matrix} 5 & 0 \\ 2 & 3 \\ 1 & -1 \end{matrix} \quad = 60+0+0-(0-0-0)=60$$

$$Or\ directly \quad 5*3*4=60$$

F-6: If each element of a row of a matrix is multiplied by a constant C and the results added to a different row, the determinant is not changed.

EX-14 **Show that** $|A|=|B| \quad if \quad A=\begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{bmatrix}$ **and** B **is the matrix resultant from multiplying row (*1*) by** 2 **and adding to row (*3*).**

i.e. $B=\begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{bmatrix}$

Sol.-

$$|A|=29 \quad from\ ex\text{-}10$$

$$|B|=\left|\begin{matrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{matrix}\right|\begin{matrix} 2 & 1 \\ -1 & 0 \\ 5 & 0 \end{matrix} \quad = 0+15+0-(0-0-14)=29$$

$$\therefore \ |A|=|B|$$

EX-15 **Find** $\begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$

Sol.-

$$\xrightarrow{-2R_1+R_2}\begin{vmatrix}1 & -2 & 3 & 1\\ 2 & 1 & 0 & 2\\ -1 & 2 & 1 & -2\\ 0 & 1 & 2 & 1\end{vmatrix}$$

Expanding the determinant by using the first column.

$$\Rightarrow 1\cdot\begin{vmatrix}5 & 2 & 0\\ 0 & 4 & -1\\ 1 & -6 & 1\end{vmatrix}\begin{matrix}5 & 2\\ 0 & 4\\ 1 & -6\end{matrix} = 20+6+0-(0-10+0)=36$$

Linear Equations

There are many methods to solve a system of linear equations:

$AX=B$

I) <u>Row Reduction method</u> It is often possible to transform the linear equations step by step into an equivalent system of equations that is so simple it can be solved by inspection.

We start with $n\times(n+1)$ matrix $[A\vdots B]$ whose first n columns are the columns of A and whose last column is B. We are going to transform this augmented matrix with a sequence of elementary row operations into $[I\vdots S]$ where S is the solution of X.

EX-16 **Solve the following linear equation:**

$$\begin{aligned}2x+3y-4z&=-3\\ x+2y+3z&=3\\ 3x-y-z&=6\end{aligned}$$

Sol.

$$AX=B \quad where \quad A=\begin{bmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad B=\begin{bmatrix}-3\\ 3\\ 6\end{bmatrix}$$

$$[A\vdots B]=\begin{bmatrix}2 & 3 & -4 & \vdots -3\\ 1 & 2 & 3 & \vdots\ 3\\ 3 & -1 & -1 & \vdots\ 6\end{bmatrix}\xrightarrow[-3R_2+R_3]{-2R_2+R_1}\begin{bmatrix}0 & -1 & -10 & \vdots -9\\ 1 & 2 & 3 & \vdots\ 3\\ 0 & -7 & -10 & \vdots -3\end{bmatrix}$$

$$\xrightarrow[R_1\ and\ R_2]{inter\ change}\begin{bmatrix}1 & 2 & 3 & \vdots\ 3\\ 0 & -1 & -10 & \vdots -9\\ 0 & -7 & -10 & \vdots -3\end{bmatrix}\xrightarrow[-7R_2+R_3]{2R_2+R_1}\begin{bmatrix}1 & 0 & -17 & \vdots -15\\ 0 & -1 & -10 & \vdots -9\\ 0 & 0 & 60 & \vdots\ 60\end{bmatrix}$$

$$\xrightarrow[\frac{1}{6}R_3+R_2]{\frac{17}{60}R_3+R_1}\begin{bmatrix}1 & 0 & 0 & \vdots\ 2\\ 0 & -1 & 0 & \vdots\ 1\\ 0 & 0 & 60 & \vdots\ 60\end{bmatrix}\xrightarrow[R_3*\frac{1}{60}]{R_2*(-1)}\begin{bmatrix}1 & 0 & 0 & \vdots\ 2\\ 0 & 1 & 0 & \vdots -1\\ 0 & 0 & 1 & \vdots\ 1\end{bmatrix}=[I\vdots S]$$

$$\textit{Hence}\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{x=2\ ,\ y=-1\ ,\ z=1}$$

II) <u>Cramer's Rule</u> When the determinant of the coefficient matrix A of the system $AX=B$ is not zero (i.e. $|A| \neq 0$) the system has a unique solution that it may be found from the formulas:

$$X_i = \frac{|A_i|}{|A|}$$

Where $|A_i|$ is the determinant of the matrix, comes from replacing the $i\ th$ column in A by the column of constant B.

<u>EX-17</u> **Resolve example 16 using Cramer's rule:**

<u>Sol.</u>

$$|A| = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix}\begin{matrix} 2 & 3 \\ 1 & 2 \\ 3 & -1 \end{matrix} = -4+27+4-(-24-6-3) = 60$$

$$|A_1| = \begin{vmatrix} -3 & 3 & -4 \\ 3 & 2 & 3 \\ 6 & -1 & -1 \end{vmatrix}\begin{matrix} -3 & 3 \\ 3 & 2 \\ 6 & -1 \end{matrix} = 6+54+12-(-48+9-9) = 120$$

$$|A_2| = \begin{vmatrix} 2 & -3 & -4 \\ 1 & 3 & 3 \\ 3 & 6 & -1 \end{vmatrix}\begin{matrix} 2 & -3 \\ 1 & 3 \\ 3 & 6 \end{matrix} = -6-27-24-(-36+36+3) = -60$$

$$|A_3| = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 2 & 3 \\ 3 & -1 & 6 \end{vmatrix}\begin{matrix} 2 & 3 \\ 1 & 2 \\ 3 & -1 \end{matrix} = 24+27+3-(-18-6+18) = 60$$

$$\therefore\quad x = \frac{|A_1|}{|A|} = \frac{120}{60} \Rightarrow x = 2$$

$$y = \frac{|A_2|}{|A|} = \frac{-60}{60} \Rightarrow y = -1$$

$$z = \frac{|A_3|}{|A|} = \frac{60}{60} \Rightarrow z = 1$$

The same result in ex-16.

Problems – 8

1) Let $A=\begin{bmatrix}1 & 2\\ 0 & 4\end{bmatrix}$, $B=\begin{bmatrix}1 & 2 & 3\\ -1 & 4 & -2\end{bmatrix}$, $C=\begin{bmatrix}3 & 1\\ 4 & -1\\ 0 & 2\end{bmatrix}$,

$D=\begin{bmatrix}1 & 0 & 4\\ 0 & 1 & 2\\ 0 & -1 & 1\end{bmatrix}$, $E=\begin{bmatrix}3 & -1\\ 4 & 2\end{bmatrix}$. **Find:-**

a) AB *b) DC* *c) (D+I)C* *d) DC+C* *e) DCB*
f) EI *g) 3A+E* *h) -5E+A* *i) E(2B)*

$$\left(ans.:\ a)\begin{bmatrix}-1 & 10 & -1\\ -4 & 16 & -8\end{bmatrix}\ b)\begin{bmatrix}3 & 9\\ 4 & 3\\ -4 & 3\end{bmatrix}\ c,d)\begin{bmatrix}6 & 10\\ 8 & 2\\ -4 & 5\end{bmatrix}\ e)\begin{bmatrix}-6 & 42 & -9\\ 1 & 20 & 6\\ -7 & 4 & -18\end{bmatrix}\right.$$
$$\left. f)\begin{bmatrix}3 & -1\\ 4 & 2\end{bmatrix}\ g)\begin{bmatrix}6 & 5\\ 4 & 14\end{bmatrix}\ h)\begin{bmatrix}-14 & 7\\ -20 & -6\end{bmatrix}\ i)\begin{bmatrix}8 & 4 & 22\\ 4 & 32 & 16\end{bmatrix}\right)$$

2) Find the value of x **:-**

$$\begin{bmatrix}x & 4 & 1\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 0 & 2\\ 0 & 2 & 4\end{bmatrix}\begin{bmatrix}x\\ -7\\ 5/4\end{bmatrix}=0$$

$$\left(ans.:\ x=\frac{1}{2}\quad or\quad x=1\right)$$

3) Find v **and** w **if:** $\begin{bmatrix}5 & w\end{bmatrix}=v\begin{bmatrix}-2 & 1\end{bmatrix}$.

$$\left(ans.:\ w=v=-\frac{5}{2}\right)$$

4) Let $A=\begin{bmatrix}1 & -1 & 2\\ 0 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}0 & 2\\ -1 & 3\\ 5 & -2\end{bmatrix}$, **Find:-**

$a)\ 2A+B'$ $b)\ B'A'-I$

$$\left(ans.:a)\begin{bmatrix}2 & -3 & 9\\ 2 & 5 & 6\end{bmatrix}\ b)\begin{bmatrix}10 & 19\\ -5 & -6\end{bmatrix}\right)$$

5) Let $A=\begin{bmatrix}3 & 0 & 1\\ 0 & 1 & 2\\ -1 & 1 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & -2 & 0\\ -1 & 3 & 2\end{bmatrix}$, **Find:-**

$(2A-I)B'$ and show that $(AB)' = B'A'$

$$\left(ans.: \begin{bmatrix} 5 & -1 \\ -2 & 11 \\ -6 & 26 \end{bmatrix} \right)$$

6) **For what value of** x **will:** $\begin{vmatrix} x & x & 1 \\ 2 & 0 & 5 \\ 6 & 7 & 1 \end{vmatrix} = 0$ **?**

$$\left(ans.: \ x = 2 \right)$$

7) **Let** A **be an arbitrary 3 by 3 matrix and let** R_{12} **be the matrix obtained from the 3 by 3 identity matrix by interchanging row 1 and 2 :** $R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$**. a) Compute** $R_{12}A$ **and show that you would get the same result by interchanging rows 1 and 2 of** A**. b) Compute** AR_{12} **and show that the result is that you would get by interchanging column 1 and 2 of** A**.**

$$\left(ans.: \ a) \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \ b) \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} \right)$$

8) **Solve the following determinants:-**

$$a) \begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix} \quad b) \begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix} \quad c) \begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$d) \begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} \quad e) \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix} \quad f) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$$g) \begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & - & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix} \quad h) \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$\left(\begin{matrix} ans.: \ a)-5 & b)\,0 & c)-7 & d)\,6 \\ e)\,38 & f)\,1 & g)\,2 & h)-1 \end{matrix} \right)$$

9) Solve the following system of equations:-

a) $x + 8y = 4$
$3x - y = -13$

b) $2x + 3y = 5$
$3x - y = 2$

c) $x + y + z = 2$
$2x - y + z = 0$
$x + 2y - z = 4$

d) $2x + y - z = 2$
$x - y + z = 7$
$2x + 2y + z = 4$

e) $2x - 4y = 6$
$x + y + z = 1$
$5y + 7z = 10$

f) $x - z = 3$
$2y - 2z = 2$
$2x + z = 3$

g) $x_1 + x_2 - x_3 + x_4 = 2$
$x_1 - x_2 + x_3 + x_4 = -1$
$x_1 + x_2 + x_3 - x_4 = 2$
$x_1 + x_3 + x_4 = -1$

h) $2x - 3y + 4z = -19$
$6x + 4y - 2z = 8$
$x + 5y + 4z = 23$

(ans.: a) $x = -4 , y = 1$ b) $x = y = 1$
c) $x = \frac{6}{7} , y = \frac{10}{7} , z = -\frac{2}{7}$ d) $x = 3 , y = -2 , z = 2$
e) $x = 0 , y = -\frac{3}{2} , z = \frac{5}{2}$ f) $x = 2 , y = 0 , z = -1$
g) $x_1 = 2 , x_2 = 0 , x_3 = x_4 = -\frac{3}{2}$ h) $x = -2 , y = 5 , z = 0$)

Chapter nine

Complex numbers

If the imaginary unit i (where $i^2 = -1$) is combine with two real numbers α, β by the processes of addition and multiplication, we obtain a complex number $\alpha + i\beta$. If $\alpha = 0$, the number is said to be purely imaginary, if $\beta = 0$ it is of course real. Zero is the only number which is at once real and imaginary.

Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

i.e. $\alpha_1 + i\beta_1 = \alpha_2 + i\beta_2 \quad \Leftrightarrow \quad \alpha_1 = \alpha_2 \quad and \quad \beta_1 = \beta_2$

Assuming that the ordinary rules of arithmetic apply to complex numbers, we find indeed:-

1. $(\alpha_1 + i\beta_1) \mp (\alpha_2 + i\beta_2) = (\alpha_1 \mp \alpha_2) + i(\beta_1 \mp \beta_2)$
2. $(\alpha_1 + i\beta_1)(\alpha_2 + i\beta_2) = (\alpha_1\alpha_2 - \beta_1\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1)$
   where $i^2 = -1$
3. $\dfrac{\alpha_1 + i\beta_1}{\alpha_2 + i\beta_2} * \dfrac{\alpha_2 - i\beta_2}{\alpha_2 - i\beta_2} = \dfrac{\alpha_1\alpha_2 + \beta_1\beta_2}{\alpha_2^2 + \beta_2^2} + i\,\dfrac{\alpha_2\beta_1 + \alpha_1\beta_2}{\alpha_2^2 + i\beta_2^2}$

The real number $\alpha_2 - i\beta_2$ that is used as multiplier to clear the i out of the denominator is called the complex conjugate of $\alpha_2 + i\beta_2$. It is customary to use $\bar{z}$ to denote the complex conjugate of z, thus $z = \alpha + i\beta$ and $\bar{z} = \alpha - i\beta$.

We note that i^n has only four possible values 1 , i , -1 , $-i$. They correspond to values of n which divided by 4 leave the reminders 0 , 1 , 2 , 3 .

***<u>EX-1</u>* – Find the values of :**

$1)\ (1+2i)^3 \qquad 2)\ \dfrac{5}{-3+4i} \qquad 3)\ \left(\dfrac{2+i}{3-2i}\right)^2$

***<u>Sol.</u>* –**

$$1)\ (1+2i)^3 = 1+6i+12i^2+8i^3 = 1+6i-12-8i = -11-2i$$

$$2)\ \frac{5}{-3+4i} * \frac{-3-4i}{-3-4i} = \frac{-15-20i}{9+16} = -\frac{3}{5} - i\frac{4}{5}$$

$$3)\ \left(\frac{2+i}{3-2i} * \frac{3+2i}{3+2i}\right)^2 = \left(\frac{6+7i-2}{9+4}\right)^2 = \left(\frac{4+7i}{13}\right)^2$$

$$= \frac{16+56i-49}{169} = -\frac{33}{169} + \frac{56}{169}i$$

EX-2- **If** *z=x+iy* **where** *x* **and** *y* **are real, find the real and imaginary parts of:-**

$1)\ z^4$ $\quad 2)\ \frac{1}{z}$ $\quad 3)\ \frac{z-1}{z+1}$ $\quad 4)\ \frac{1}{z^2}$

Sol.-

$$1)\ z^4 = (x+iy)^4 = x^4 + 4x^3(iy) + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$
$$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)$$

$$2)\ \frac{1}{z} = \frac{1}{x+iy} * \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$3)\ \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} * \frac{(x+1)-iy}{(x+1)-iy} = \frac{x^2-1-2iy+y^2}{(x+1)^2+y^2}$$
$$= \frac{x^2+y^2-1}{(x+1)^2+y^2} - i\frac{2y}{(x+1)^2+y^2}$$

$$4)\ \frac{1}{z^2} = \frac{1}{(x+iy)^2} = \frac{1}{x^2-y^2+2xyi} * \frac{x^2-y^2-2xyi}{x^2-y^2-2xyi}$$
$$= \frac{x^2-y^2-2xyi}{(x^2-y^2)^2+4x^2y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} - i\frac{2xy}{(x^2+y^2)^2}$$

EX-3- **Show that** $\left(\frac{-1\mp i\sqrt{3}}{2}\right)^3 = 1$ **for all combination of signs.**

Sol.-

$$L.H.S.=\left(\frac{-1\mp i\sqrt{3}}{2}\right)^3=\frac{1}{8}\left[(-1)^3+3(-1)^2(\mp i\sqrt{3})+3(-1)(\mp i\sqrt{3})^2+(\mp i\sqrt{3})^3\right]$$

$$=\frac{1}{8}\left[-1\mp \cancel{i3\sqrt{3}}+9\pm \cancel{i3\sqrt{3}}\right]=1=R.H.S.$$

<u>*EX-4*</u>- Solve the following equation for the real numbers x and y.

$$(3+4i)^2-2(x-iy)=x+iy$$

<u>*Sol.*</u>-

$$9+24i+16i^2=2x-2iy+x+iy$$

$$-7+24i=3x-iy \quad \Rightarrow \quad -7=3x \quad \Rightarrow \quad x=-\frac{7}{3}$$

$$\Rightarrow \quad 24=-y \quad \Rightarrow \quad y=-24$$

<u>Argrand Diagrams:-</u> There are two geometric representation of the complex number $z=x+iy$:-

a) as the point *P(x,y)* in the *xy*-plane , and

b) as the vector $\overrightarrow{op}$ from the origin to *P*.

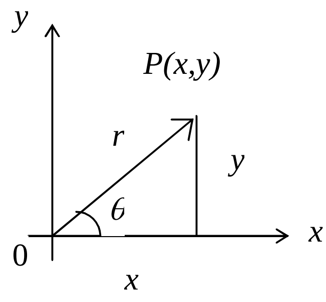
In each representation, the *x*-axis is called the real axis and the *y*-axis is the imaginary axis, as following figure.

In terms of the polar coordinates of x and y , we have:-

$$x=r\cos\theta \quad , \quad y=r\sin\theta \quad , \quad \tan\theta=\frac{y}{x}$$

and $z=r(\cos\theta+i\sin\theta)$

(polar representation)



The length r of a vector $\overrightarrow{op}$ from the origin to *P(x,y)* is:

$$|x+iy|=\sqrt{x^2+y^2}$$

The polar angle θ is called the argument of z and is written $\theta=\arg z$

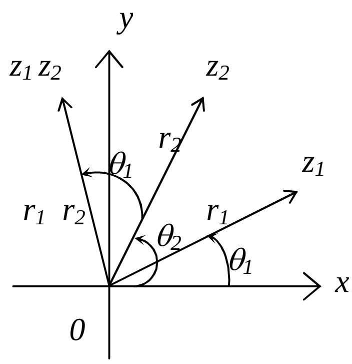
The identity $e^{i\theta} = \cos\theta + i\sin\theta$ is used for calculating products, quotients, powers, and roots of complex numbers. Then $z = re^{i\theta}$ exponential representation.

a) Product: To multiply two complex numbers (figure below):

$z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ so that $\begin{matrix} |z_1| = r_1 & , & \arg z_1 = \theta_1 \\ |z_2| = r_2 & , & \arg z_2 = \theta_2 \end{matrix}$

Then $z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

hence $\begin{matrix} |z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2| \\ \arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2 \end{matrix}$



b) Quotients: $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

hence $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$

<u>*EX-5*</u>- Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$ find:

1) the exponential representation for z_1 and z_2.
2) the values of $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in exponential and polar representations.

<u>*Sol.*</u>-

1) $z_1 = 1 + i \Rightarrow x_1 = 1 \;,\; y_1 = 1 \Rightarrow r_1 = \sqrt{x_1^2 + y_1^2} = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \dfrac{y}{x} \Rightarrow \theta_1 = \tan^{-1}\dfrac{y_1}{x_1} = \tan^{-1} 1 = \dfrac{\pi}{4}$ $\quad \therefore z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$z_2 = \sqrt{3} - i \Rightarrow x_2 = \sqrt{3} \;,\; y_2 = -1 \Rightarrow r_2 = \sqrt{x_2^2 + y_2^2} = \sqrt{3+1} = 2$

$\Rightarrow \theta_2 = \tan^{-1}\dfrac{y_2}{x_2} = \tan^{-1}\dfrac{-1}{\sqrt{3}} = -\dfrac{\pi}{6}$ $\quad \therefore z_2 = 2\, e^{-i\frac{\pi}{6}}$

$$2)\quad z_1 z_2 = \sqrt{2}\, e^{i\frac{\pi}{4}} \cdot 2\, e^{-i\frac{\pi}{6}} = 2\sqrt{2} e^{i\frac{\pi}{12}} \qquad \textit{exponential representation}$$

$$r = 2\sqrt{2} \quad , \quad \theta = \frac{\pi}{12} \quad \Rightarrow$$

$$z_1 z_2 = 2\sqrt{2}\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12} \right) \qquad \textit{polar representation}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\frac{\pi}{4}}}{2\, e^{-i\frac{\pi}{6}}} = \frac{1}{\sqrt{2}} e^{i\frac{5}{12}\pi} \qquad \textit{exponential representation}$$

$$r = \frac{1}{\sqrt{2}} \quad , \quad \theta = \frac{5}{12}\pi \quad \Rightarrow$$

$$\frac{z_1}{z_2} = \frac{1}{\sqrt{2}}\left(\cos(\frac{5}{12}\pi) + i \sin(\frac{5}{12}\pi) \right) \qquad \textit{polar representation}$$

c) **Powers: If n is a positive integer, then:**

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} \qquad \textit{hence } \left|z^n\right| = r^n \quad \textit{and} \quad \arg z^n = n\theta$$

$$\textit{DeMoivres Theorem :} \quad (\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

EX-6- **Find:** $\left(\sqrt{3} - i\right)^{10}$

Sol.-

$$\sqrt{3} - i \;\underset{y=-1}{\overset{x=\sqrt{3}}{\Longrightarrow}}\; r = \sqrt{3+1} = 2 \quad \textit{and} \quad \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-1}{\sqrt{3}} = -\frac{\pi}{6}$$

$$\sqrt{3} - i = 2\left(\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}) \right) = 2\left(\cos\frac{\pi}{6} - i \sin\frac{\pi}{6} \right)$$

$$\left(\sqrt{3} - i\right)^{10} = 2^{10}\left(\cos 10\frac{\pi}{6} - i \sin 10\frac{\pi}{6} \right) = 512 + 512\sqrt{3} i$$

d) **Roots: If $z = re^{i\theta}$ is a complex number different from zero and n is a positive integer, then there are precisely n different complex numbers $w_0, w_1, w_2, \ldots\ldots, w_{n-1}$, that are *nth* roots of z given by:**

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r}\, e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)} \quad , \quad k = 0,1,2,\ldots\ldots\ldots, n-1$$

EX-7- **Find the four forth roots of (-16)**

Sol.-

$$z=-16 \Rightarrow r=\sqrt{(-16)^2+0}=16 \quad \& \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{0}{-16}=\pi$$

$$\sqrt[4]{-16}=\sqrt[4]{16}\, e^{i(\frac{\pi}{4}+k\frac{2\pi}{4})}=2e^{i(\frac{\pi}{4}+\frac{\pi}{2}k)} \quad , \quad k=0,1,2,3$$

$$at \quad k=0 \Rightarrow \quad 1st\ root=w_0=2e^{i\frac{\pi}{4}}=2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}+\sqrt{2}i$$

$$at \quad k=1 \Rightarrow \quad 2nd\ root=w_1=2e^{i\frac{3\pi}{4}}=2\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-\sqrt{2}+\sqrt{2}i$$

$$at \quad k=2 \Rightarrow \quad 3rd\ root=w_2=2e^{i\frac{5\pi}{4}}=2\left(\cos\frac{5}{4}\pi+i\sin\frac{5}{4}\pi\right)=-\sqrt{2}-\sqrt{2}i$$

$$at \quad k=3 \Rightarrow \quad 4th\ root=w_3=2e^{i\frac{7}{4}\pi}=2\left(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi\right)=\sqrt{2}-\sqrt{2}i$$

EX-8- **Find the four solutions of the equation:-** $z^4-2z^2+4=0$

Sol.-

$$z^4-2z^2+4=0 \Rightarrow z^2=\frac{2\mp\sqrt{4-4*1*4}}{2*1}=1\mp\sqrt{3}i \Rightarrow z=\mp\sqrt{1\mp i\sqrt{3}}$$

$$\left\{ for \quad ax^2+bx+c=0 \quad \Rightarrow \quad x=\frac{-b\mp\sqrt{b^2-4ac}}{2a} \right\}$$

$$for \quad \sqrt{1+i\sqrt{3}} \quad \Rightarrow \quad r=\sqrt{1+3}=2 \quad and \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{\sqrt{3}}{1}=\frac{\pi}{3}$$

$$1st\ root=w_0=\sqrt{2}e^{\frac{i}{2}(\frac{\pi}{3})}=\sqrt{2}e^{i\frac{\pi}{6}}=\sqrt{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i$$

$$2nd\ root=w_1=\sqrt{2}e^{\frac{i}{2}(\frac{\pi}{3}+2\pi)}=\sqrt{2}e^{i\frac{7}{6}\pi}=\sqrt{2}\left(\cos\frac{7}{6}\pi+i\sin\frac{7}{6}\pi\right)$$

$$=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i$$

$$for \quad \sqrt{1-i\sqrt{3}} \quad \Rightarrow \quad r=\sqrt{1+3}=2 \quad and \quad \theta=\tan^{-1}\frac{y}{x}=\tan^{-1}\frac{-\sqrt{3}}{1}=-\frac{\pi}{3}$$

$$3rd\ root = w_2 = \sqrt{2}e^{\frac{i}{2}(-\frac{\pi}{3})} = \sqrt{2}e^{i(-\frac{\pi}{6})} = \sqrt{2}\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)$$

$$= \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$$

$$4th\ root = w_3 = \sqrt{2}e^{\frac{i}{2}(-\frac{\pi}{3}+2\pi)} = \sqrt{2}e^{i\frac{5}{6}\pi} = \sqrt{2}\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right)$$

$$= -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$$

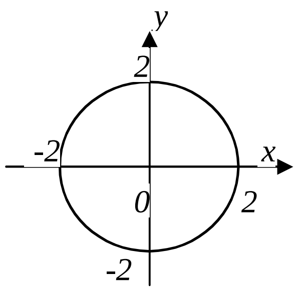
EX-9- **Graph the points $z = x + iy$ that satisfy the given conditions:-**

$1)\ |z| = 2 \qquad 2)\ |z| < 2 \qquad 3)\ |z| > 2 \qquad 4)\ |z+1| = |z-1|$

Sol.-

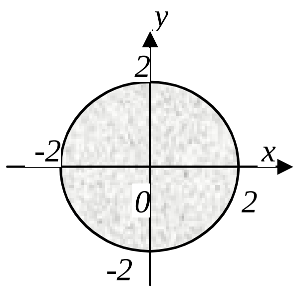
$1)\ |z| = 2 \Rightarrow \sqrt{x^2 + y^2} = 2 \Rightarrow x^2 + y^2 = 4$

The points on the circle with center at origin, and radius 2.



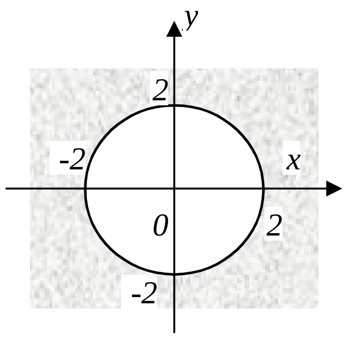
$2)\ |z| < 2 \Rightarrow \sqrt{x^2 + y^2} < 2 \Rightarrow x^2 + y^2 < 4$

The interior points of the circle with center at origin, and radius 2.



$3)\ |z| > 2 \Rightarrow \sqrt{x^2 + y^2} > 2 \Rightarrow x^2 + y^2 > 4$

The exterior points of the circle with center at origin, and radius 2.

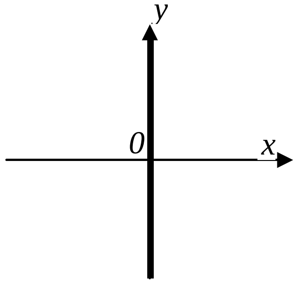


$4)\ |z+1| = |z-1| \Rightarrow |x + iy + 1| = |x + iy - 1|$

$\Rightarrow \sqrt{(x+1)^2 + y^2} = \sqrt{(x-1)^2 + y^2} \Rightarrow$

$x^2 + 2x + 1 + y^2 = x^2 - 2x + 1 + y^2 \Rightarrow x = 0$

The points on the y - axis.



Problems – 9

1) Find the values of:-

a) $(2+3i)(4-2i)$ (ans. : $14+8i$)

b) $(2-i)(-2+3i)$ (ans. : $-1+8i$)

c) $(-1-2i)(2+i)$ (ans. : $-5i$)

2) Show that $\left(\dfrac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$ **for all combination of signs.**

3) Solve the following equation for the real numbers x and y :-

$(3-2i)(x+iy) = 2(x-2iy)+2i-1$ (ans. : $x=-1$; $y=0$)

4) Show that $|\bar{z}| = |z|$.

5) Let $Re(z)$ and $Im(z)$ denote respectively the real and imaginary parts of z , show that:-

a) $z+\bar{z} = 2\,Re$

b) $z-\bar{z} = 2i\,Im(z)$

c) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,Re(z_1\,\bar{z}_2)$

6) Graph the points $z = x+iy$ that satisfy the given conditions:-

a) $|z-1| = 2$ (ans. : on the circle with center (1,0), radius 2)

b) $|z+1| = 1$ (ans. : on the circle with center (−1,0), radius 1)

c) $|z+i| = |z-1|$ (ans. : on the line $y=-x$)

7) Express the following complex number in exponential form with $r \ge o$ and $-\pi < \theta < \pi$:-

$a)\ (1+\sqrt{-3})^2$ $\qquad$ $(ans.:\ 4e^{i\frac{2}{3}\pi})$

$b)\ \dfrac{1+i}{1-i}$ $\qquad$ $(ans.:\ e^{i\frac{\pi}{2}})$

$c)\ \dfrac{1+i\sqrt{3}}{1-i\sqrt{3}}$ $\qquad$ $(ans.:\ e^{i\frac{\pi}{2}})$

$d)\ (2+3i)(1-2i)$ $\qquad$ $(ans.:\ \sqrt{65}e^{i\tan^{-1}(-0.125)})$

8) **Find the three cube roots of** *1* **.** $\qquad$ $(ans.:\ -\dfrac{1}{2}\mp i\dfrac{\sqrt{3}}{2})$

9) **Find the two square roots of** *i* **.** $\qquad$ $(ans.:\ \mp\dfrac{1}{\sqrt{2}}\mp i\dfrac{1}{\sqrt{2}})$

10) **Find the three cube roots of** *(-8i)* **.**

$(ans.:\ -2i\ ;\ \mp\sqrt{3}-i)$

11) **Find the six sixth roots of** *(64)* **.**

$(ans.:\ \mp 2\ ;\ 1\mp i\sqrt{3}\ ;\ -1\mp i\sqrt{3})$

12) **Find the six solutions of the equation:** $\qquad z^6+2z^3+2=0$

$(ans.:\ \sqrt[3]{2}\left(\cos\dfrac{2}{9}\pi\mp i\sin\dfrac{2}{9}\pi\right)\ ;$

$\sqrt[3]{2}\left(-\cos\dfrac{\pi}{9}\mp i\sin\dfrac{\pi}{9}\right)\ ;\ \sqrt[3]{2}\left(\cos\dfrac{4}{9}\pi\mp i\sin\dfrac{4}{9}\pi\right))$

13) **Find all solutions of the equation:** $\qquad x^4+4z^2+16=0$

$(ans.:\ 1\mp i\sqrt{3}\ ;\ -1\mp i\sqrt{3})$

14) **Solve the equation:** $\qquad x^4+1=0$

$(ans.:\ \dfrac{1}{\sqrt{2}}\mp\dfrac{i}{\sqrt{2}}\ ;\ -\dfrac{1}{\sqrt{2}}\mp\dfrac{i}{\sqrt{2}})$